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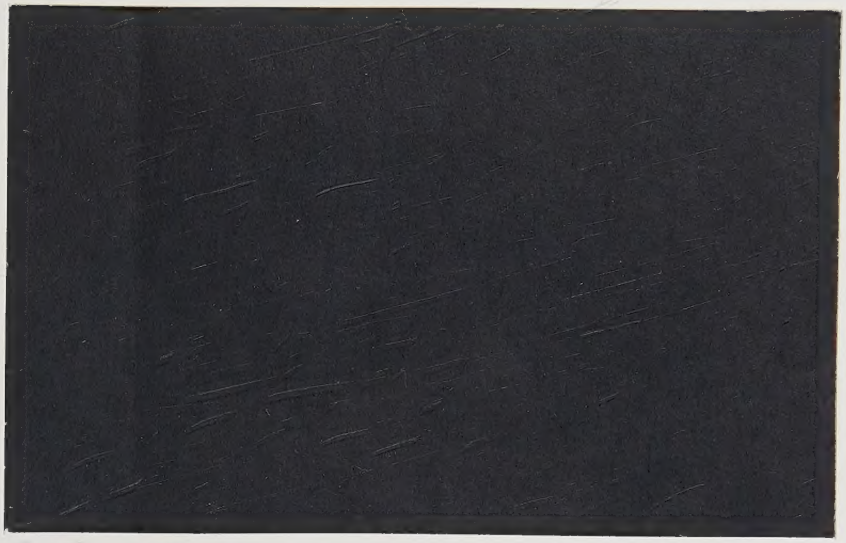

DEPARTMENT OF MATHEMATICS

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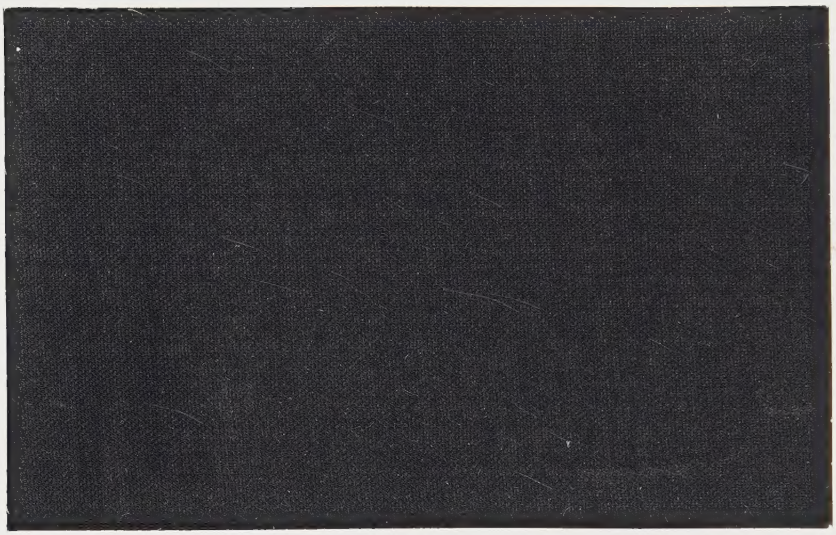

ON APPROXIMATION BY TRANSLATES AND RELATED PROBLEMS IN FUNCTION THEORY

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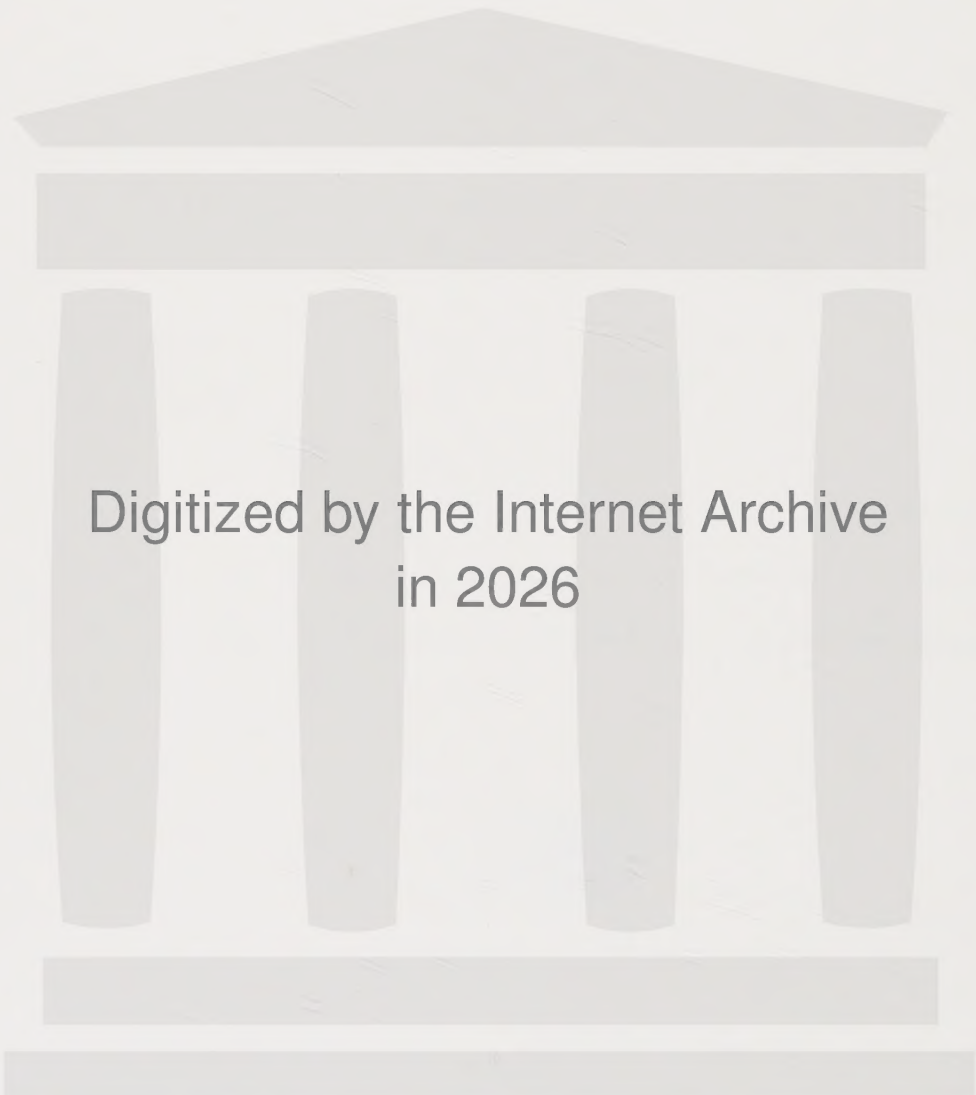
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1. Introduction.

From Wiener's approximation theorem we know that the set of finite linear combinations of translates of a function $f \in L(R)$ is dense in $L(R)$ if and only if its Fourier transform $\hat{f}$ is never zero.

What can be said if we only allow translates $f(\cdot-\lambda)$ with λ belonging to some fixed set Λ? Problems of this type have been studied by Ganelius [7], Landau [11] and Lönnroth [14], in the case that Λ has a finite point of accumulation. Edwards [5] and Lönnroth treat problems of weighted approximation by translates of functions belonging to a special class.

We will mainly be interested in sets Λ with infinity as accumulation point. Several specific approximation problems can be transformed to problems of approximation by translates. We take the Müntz-Szász approximation theorem as an example.

Given $g \in L[0,1]$, $\varepsilon > 0$ and a sequence of distinct numbers $\mu_k > -1$ such that

$$\Sigma \min \left(\frac{1}{1+\mu_k}, 1+\mu_k\right) = \infty$$

we can find finitely many numbers a_k, such that

$$\int_0^1 \left|\Sigma\, a_k x^{\mu_k} - g(x)\right| dx < \varepsilon.$$

Transforming this expression by the substitution $x = \exp(-\exp(-t))$ and putting $f(t) = \exp(-\exp(-t)-t)$ we have

$$\int_{-\infty}^{\infty} \left|\Sigma \frac{a_k}{1+\mu_k} f(t-\log(1+\mu_k)) - g(e^{-e^{-t}})e^{-e^{-t}-t}\right| dt =$$

$$= \int_{-\infty}^{\infty} \left|\Sigma\, b_k f(t-\lambda_k) - h(t)\right| dt < \varepsilon.$$

We find that approximation in L by linear combinations of $f(\cdot-\lambda_k)$ is possible if and only if

$$\Sigma \exp(-|\lambda_k|) = \infty.$$

We will consider functions f such that

$$\hat{f}(t) = O(e^{-\theta(t)}), \quad |t| \to \infty,$$

θ belonging to some specified class, and try to determine for which Λ approximation is possible. The spaces we deal with are L, L^2 and C_0.

The general idea is to derive conditions on Λ related to the behaviour as $T \to \infty$ of

$$\int_1^T \frac{\theta(t)}{t^2}\, dt.$$

Our main results on this problem are contained in section 3.1 where we assume that θ is even, convex, and that $\theta(t)/t \to \alpha \neq 0, \infty$, as $t \to \infty$. For certain classes of f we give necessary and sufficient conditions on Λ. As a special case, observing that in the example above $\hat{f}(t) = (2\pi)^{-1/2}\, \Gamma(1+it) = O(t^{1/2} \exp(-\frac{\pi}{2}|t|))$, we obtain the Müntz-Szász theorem.

To prove these theorems we need some results on zeros of functions holomorphic in a strip, which we prove in ch. 2. We consider functions satisfying

$$|F(x+iy)| \leq \exp(\rho(y)), \quad |y| < \alpha,$$

with ρ convex, $\rho(y) \to \infty$ as $|y| \to \alpha$. Our method is based on Ahlfors distortion theorem. We also state a conjecture on zeros of functions belonging to certain quasi-analytic classes.

In section 3.2 we seek other properties of f essential with respect to our problem. We show that certain irregularity conditions on $\hat{f}$ can in some cases imply good approximation properties.

In section 3.3 we turn to more general θ. Our problem has obvious connections with quasi-analyticity and we use known results from this field to give some information. In particular we discuss the case $\theta(t)/t \to 0$ as $t \to \infty$.

In the last section we first treat the case $\hat{f}(t) = O(-\alpha|t|^p)$, $p > 1$. Assuming that Λ satisfies a separation condition we obtain a rather sharp theorem. Zalik [17] has recently annonced related results for the non-separated case. From the short notice it seems that the results do not involve the same degree of precision.

Finally we look upon our problem from another side. Given a set of translation numbers Λ, is there any function $f \in L$ such that $\{f(\cdot-\lambda), \lambda \in \Lambda\}$ spans L? We show that Λ cannot be too sparse and give a related result for L^2.

Acknowledgement. I wish to thank Professor Tord Ganelius for suggesting the topic and for his support and kind interest in my work.

I also wish to thank Boel Engebrand, Kerstin Lidhag, Lena Mattsson and Ebon Norrman for their fine job in typing the manuscript.

2. Zeros of functions holomorphic in a strip.

In this chapter we first prove a theorem about real zeros of a function, holomorphic and satisfying a certain growth condition in a strip (Th 2.1.1).

Then we restate the theorem with the growth condition substituted by a condition on the Fourier transform of the function. This formulation leads us to conjecture that a more general statement holds. We give a result in this direction in theorem 2.3.1.

Finally we show that theorem 2.1.1 is best possible under certain regularity assumptions.

We let S_α be the strip $\{z = x+iy : |y| < \alpha\}$.

2.1 If $f \not\equiv 0$ is holomorphic and uniformly bounded in the strip S_α, and λ_k are real zeros of f, then, by the Blaschke condition,

$$\Sigma \exp(-\frac{\pi}{2\alpha} |\lambda_k|) < \infty .$$

We consider functions satisfying a growth condition of the form $|f(x+iy)| \leq C \cdot \exp(\rho(y))$, $|y| < \alpha$, and derive a modified Blaschke condition. In particular we are interested in finding for which ρ the condition remains unchanged. Since $\sup_x |f(x+iy)|$ is a logarithmically convex function of y, it is sufficient to consider convex ρ.

We will need some results concerning convex functions. If h is a continuous convex function defined on some interval I, then its convex conjugate, h^*, defined by $h^*(y) = \sup_{x \in I} (xy - h(x))$, is a continuous convex function on the interval where it is finite. Taking the conjugate a second time we obtain an extension of h, i.e. $h^{**}(x) = h(x)$, $x \in I$. If h is even, then h^* is also even, and if, in addition, $h(0) < 0$, then $h^* > 0$. For these and other results on convexity, see Rockafellar [16].

In this section Λ will denote a sequence of real numbers, $\{\lambda_n\}_0^\infty$.

Theorem 2.1.1. Let f be analytic in the strip S_α, let ρ be an even, convex function such that $\rho(0) < 0$ and $\rho(y) \to \infty$ as $|y| \to \alpha$, and let ρ^* be the convex conjugate of ρ.

Suppose that for some $C > 0$

$$|f(x+iy)| \leqq C \exp(\rho(y)), \quad |y| < \alpha . \tag{2.1.1}$$

Given $\lambda \in R$, define λ^* by

$$\frac{2}{\pi} \int_1^{\lambda^*} \frac{\rho^*(x)}{x^2}\, dx = |\lambda| . \tag{2.1.2}$$

If Λ is a sequence of real zeros of f (counting multiplicities), and

$$\sum_{\lambda_n \in \Lambda} \frac{1}{\lambda_n^*} = \infty \tag{2.1.3}$$

then $f \equiv 0$.

In particular, if $\int_0^\alpha \log^+ \rho(y)dy < \infty$ and $\sum_{\lambda_n \in \Lambda} \exp(-\frac{\pi}{2\alpha} |\lambda_n|) = \infty$, then $f \equiv 0$.

Proof. It is no restriction to assume that ρ is twice continuously differentiable with $\rho'' > 0$ since such a function easily can be found between $\rho(x)$ and $\rho(x) - 1$, and the corresponding series $\Sigma\, 1/\lambda_n^*$ will converge or diverge together with the original series.

It is clear that ρ^* is an even function defined on R. From the definition we have

$$\rho^*(t) = st - \rho(s), \text{ where } \rho'(s) = t. \tag{2.1.4}$$

Since

$$\frac{dt}{ds} = \rho''(s) > 0 \tag{2.1.5}$$

we have that s and ρ^* are continuously differentiable functions of t.

Differentiating 2.1.4 yields

$$\frac{d}{dt}\rho^*(t) = s. \tag{2.1.6}$$

Define

$$\theta(t) = \rho^*(t). \tag{2.1.7}$$

Then $\rho(\theta'(t)) = \rho(s) = st - \rho^*(t) = t\theta'(t) - \theta(t)$.

Hence, by the hypothesis,

$$|f(x \pm i\theta'(t))| \leq C \exp(t\theta'(t) - \theta(t)) \tag{2.1.8}$$

independently of x. By convexity the right side increases with t.

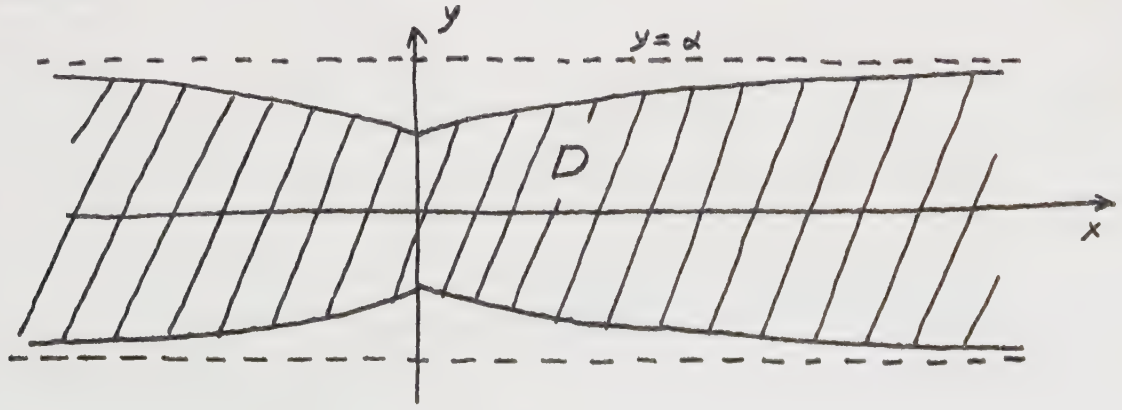
We also note that $\theta'(t) > 0$ for $t > 0$ and $\theta(t)/t \to \alpha$ as $t \to \infty$.

The images of $[0,\infty[$ under the four mappings

$$t \rightsquigarrow \pm \frac{2}{\pi} \int_0^t \theta'(e^u)du \pm i\theta'(e^t) \tag{2.1.9}$$

together enclose a region which we denote by D.

Figure 1



Let ϕ be the conformal mapping of the strip $S_{\pi/2}$ onto D, satisfying $\phi(0) = 0$, $\phi'(0) > 0$. This will ensure that ϕ preserves symmetry with respect to the axes.

Let $\psi(u)$ be half the length of the intersection between the line $x = u$ and D. For $\mu \in R$, put $\lambda = \phi(\mu)$. By Ahlfors' distortion theorem, [1], there is a constant k, independent of μ, such that

$$(2.1.10) \qquad \left|\mu - \frac{\pi}{2} \int_0^{\lambda} \frac{dt}{\psi(t)}\right| < k .$$

We assume $\lambda > 0$, which is no restriction by reasons of symmetry.

Define

$$(2.1.11) \qquad \varphi(\lambda) = \frac{\pi}{2} \int_0^{\lambda} \frac{dt}{\psi(t)} .$$

Substituting $t = \frac{2}{\pi} \int_0^u \theta'(e^x)dx$ in the integral and observing that, by the construction of D, $\psi(t) = \theta'(e^u)$, we have $\frac{\pi}{2} \int_0^{t(u)} \frac{dt}{\psi(t)} = \int_0^u ds = u$. Hence

$$(2.1.12) \qquad \varphi^{-1}(u) = t(u) = \frac{2}{\pi} \int_0^u \theta'(e^x)dx .$$

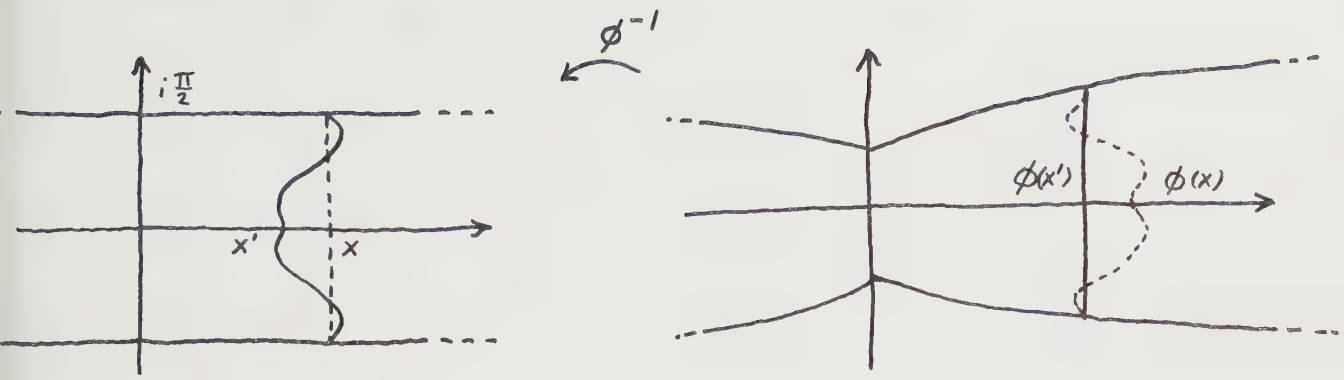
We can write (2.1.10) as

$$(2.1.13) \qquad |\phi^{-1}(\lambda) - \varphi(\lambda)| < k, \quad \forall \lambda \in R .$$

Put $h(z) = f(\phi(z))$. The function h is holomorphic in $S_{\pi/2}$, has real zeros at the points $\phi^{-1}(\lambda_n) = \mu_n$, and can be continuously extended to $\bar{S}_{\pi/2}$ since all boundary points of D are simple.

We wish to estimate h on the boundary. For $x \geq 0$ we define x' by $\phi(x') = \mathrm{Re}\ \phi(x + i\frac{\pi}{2})$.

Figure 2



The distortion theorem applied to ϕ^{-1} shows that $|x'-x| < k_1$, where k_1 is independent of x, since $|x'-x|$ is less than the horizontal distortion of the segment $\{z: \mathrm{Re}(z) = \phi(x')\} \cap D$.

Now, introduce u by $\varphi^{-1}(u) = \phi(x')$. Then

$$(2.1.14) \qquad h(x+i\tfrac{\pi}{2}) = f(\phi(x') + i\psi(\phi(x'))) = f(\tfrac{2}{\pi}\int_0^u \theta'(e^t)dt + i\theta'(e^u)).$$

But $|u-x| \leqq |u-x'| + |x-x'| = |\varphi(\phi(x'))- \phi^{-1}(\phi(x'))| + |x-x'| \leqq k+k_1 = k_2$.

Hence we obtain the following estimate by (2.1.8) and (2.1.14);

$$(2.1.15) \qquad |h(x+i\tfrac{\pi}{2})| \leqq C \exp(e^{x+k_2}\theta'(e^{x+k_2}) - \theta(e^{x+k_2})), \quad x \geqq 0.$$

We claim that, if $h(0) \neq 0$ and $0 < \beta < \pi/2$,

$$(2.1.16) \qquad \log|h(0)| + \Sigma \log\left|\frac{\exp(\pi\mu_n/2\beta)+1}{\exp(\pi\mu_n/2\beta)-1}\right| \leqq$$

$$\leqq \frac{1}{2\beta}\int_{-\infty}^{\infty} \frac{\log|h(x+i\beta)h(x-i\beta)|dx}{\exp(\pi x/2\beta)+\exp(-\pi x/2\beta)}.$$

This inequality is obtained by a transformation to the strip S_β of the well known inequality for holomorphic functions, uniformly bounded in the unit disc, U;

$$\log |F(0)| + \Sigma \log(\frac{1}{r_k}) \leqq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |F^*(e^{i\theta})| d\theta ,$$

where F^* is the radial limit function of F and $\{r_k e^{i\theta_k}\}$ are the zeros of F in U.

We have to show that h is uniformly bounded in S_β. This follows from

$$|\mathrm{Im}\ \phi(x+iy)| \leqq \frac{2\alpha}{\pi} |y| . \tag{2.1.17}$$

To see this, observe that $\mathrm{Im}\ \phi(z)$ is harmonic in D, continuous and uniformly bounded in $\bar{D}$, and $\mathrm{Im}\ \phi(z) - \frac{2\alpha}{\pi} y \leqq 0$ for $y = 0$ and $y = \pi/2$. By a Phragmén-Lindelöf argument the inequality follows for $y \geqq 0$ which is sufficient by symmetry.

Let us assume that $f \not\equiv 0$. There is no loss of generality in taking $f(0) \neq 0$. If we use the symmetry of h, and observe that (2.1.15) is valid also when the left member is replaced by $|h(x+i\beta)|$, we find that for $\beta > \pi/4$, say, the right member of (2.1.16) is not larger than

$$I = C \int_0^\infty \frac{e^x \theta'(e^x) - \theta(e^x)}{e^x} dx$$

for some $C > 0$. But

$$I = C \int_1^\infty \frac{t\theta'(t) - \theta(t)}{t^2} dt = C \left[\frac{\theta(t)}{t}\right]_1^\infty < C\alpha .$$

Since $\log \left|\frac{\exp(\pi\mu_n/2\beta) + 1}{\exp(\pi\mu_n/2\beta) - 1}\right| \geqq 2 \exp(-\pi|\mu_n|/2\beta)$, it follows, by letting β tend to $\pi/2$ in (2.1.16), that

$$\Sigma \exp(-|\mu_n|) < \infty . \tag{2.1.18}$$

But

$$\frac{2}{\pi}\int_0^{\log(\lambda_n^*)} \theta'(e^x)dx = \frac{2}{\pi}\int_1^{\lambda_n^*} \frac{\theta'(x)}{x}\,dx = \frac{2}{\pi}\left[\frac{\theta(x)}{x}\right]_1^{\lambda_n^*} + \frac{2}{\pi}\int_1^{\lambda_n^*} \frac{\theta(x)}{x^2}\,dx .$$

Hence

$$\varphi^{-1}(\log \lambda_n^*) = O(1) + |\lambda_n| .$$

Consequently,

$$\log \lambda_n^* = \varphi(O(1) + |\lambda_n|) = \varphi(|\lambda_n|) + O(1) = \tag{2.1.19}$$

$$= \phi^{-1}(|\lambda_n|) + O(1) = |\mu_n| + O(1),$$

which by (2.1.18) contradicts the assumption, (2.1.3).

Hence $f \equiv 0$ and the theorem is proved, except for the last statement, which we prove in section 2.2.

We now restate the first part of theorem 2.1.1.

Theorem 2.1.2. Let θ be an even, positive, convex function such that $\theta(t)/t \to \alpha > 0$ as $t \to \infty$.

Let $g \in L$ and suppose $g(t) = O(e^{-\theta(t)})$, $(|t| \to \infty)$.

Let f be the Fourier transform of g.

Given $\lambda \in R$ define λ^* by

$$|\lambda| = \frac{2}{\pi}\int_1^{\lambda_n^*} \frac{\theta(t)}{t^2}\,dt .$$

If Λ is a sequence of real zeros of f (counting multiplicities) and $\sum_{\lambda_n \in \Lambda} 1/\lambda_n^* = \infty$, then $f \equiv 0$.

Proof. We can write

$$|g(t)| \leqq \frac{C}{1+t^2} \exp(-\theta(t) + \log(1+t^2)) .$$

Choose M such that $\theta(t) - \log(1+t^2) + M$ is positive for all t, and let θ_1 be the largest convex minorant to $\theta(t) - \log(1+t^2) + M$.

By hypothesis

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itx} g(x)dx .$$

Since g is exponentially decreasing this integral defines an analytic extension of f to the strip S_α. We denote this extension by the same letter, f.

Hence, for $|y| < \alpha$,

$$|f(x+iy)| \leqq C_0 \int_0^{\infty} \exp(t|y| - \theta_1(t)) \frac{dt}{1+t^2} \leqq C_1 \exp(\theta_1^*(|y|)).$$

Let us define λ_n^+ by

$$|\lambda_n| = \frac{2}{\pi} \int_1^{\lambda_n^+} \frac{\theta_1(t)}{t^2} dt .$$

By theorem 2.1.1 and since $\theta_1^{**} = \theta_1$, $f \equiv 0$ follows from $\Sigma\, 1/\lambda_n^+ = \infty$.

We show that there are positive constants, c_1 and c_2, such that $c_1\lambda_n^* < \lambda_n^+ < c_2\lambda_n^*$. Since, by hypothesis, $\Sigma\, 1/\lambda_n^* = \infty$ it then follows that $f \equiv 0$.

We observe that $\theta_1(t) = \theta(t) - \log(1+t^2) + M$ for t sufficiently large. From the definitions of λ_n^* and λ_n^+ we have

$$0 = - \int_1^{\lambda^*} \frac{\theta_1(t) - \theta(t)}{t^2} dt + \int_{\lambda^*}^{\lambda^+} \frac{\theta_1(t)}{t^2} dt .$$

Since, for some $\gamma > 0$, $\gamma|t| \leqq \theta_1(t) \leqq \alpha|t| + \theta_1(0)$ we obtain

$$k_1 < \log(\frac{\lambda^+}{\lambda^*}) < k_2$$

which proves the assertion.

2.2. Some remarks on $\Sigma\, 1/\lambda_n^*$.

Let ρ be as in theorem 2.1.2. It is clear that, for $x \geqq 0$,

$$\rho^*(x) = \alpha x - \beta(x) + \rho^*(0) \tag{2.2.1}$$

where β is a concave, nonnegative function.

As $x \to \infty$, $\frac{d}{dx}\rho^*(x) \to \alpha$ by (2.1.6), hence $\beta'(x) \to 0$ and $\beta(x)/x \to 0$.

By the definition of λ_n^*,

$$|\lambda_n| = \frac{2\alpha}{\pi}\log(\lambda_n^*) - \frac{2}{\pi}\int_1^{\lambda_n^*} \frac{\beta(x)}{x^2}\,dx\,.$$

Thus, if the integral $\int_1^\infty \frac{\beta(x)}{x^2}\,dx$ is convergent, then the two series $\Sigma\, 1/\lambda_n^*$ and $\Sigma \exp(-\frac{\pi}{2\alpha}|\lambda_n|)]$ converge or diverge together.

On the other hand, if the integral diverges,

$$1/\lambda_n^* = o[\exp(-\tfrac{\pi}{2\alpha}|\lambda_n|)] \quad \text{as} \quad |\lambda_n| \to \infty\,.$$

It is then possible to find $\Lambda = \{\lambda_n\}_1^\infty$ such that $\Sigma\, 1/\lambda_n^* < \infty$ but $\Sigma \exp(-\frac{\pi}{2\alpha}|\lambda_n|) = \infty$. To see this, choose $\nu_k > 0$ for $k = 1,2,\ldots$ such that

$$1/\nu_k^* = (1/k^2)\exp(-\tfrac{\pi}{2\alpha}\nu_k).$$

Let Λ be the sequence where each ν_k is repeated $[\exp(\frac{\pi}{2\alpha}\nu_k)]$ times.

Then

$$\sum_{n=1}^{\infty} 1/\lambda_n^* \leqq \sum_{k=1}^{\infty} (1/\nu_k^*)\exp(\frac{\pi}{2\alpha}\nu_k) = \sum_{k=1}^{\infty} 1/k^2 < \infty ,$$

and

$$\sum_{n=1}^{\infty} \exp(-\frac{\pi}{2\alpha}|\lambda_n|) \geqq \sum_{k=1}^{\infty} (1 - e^{-\frac{\pi}{2\alpha}\nu_k}) = \infty .$$

Hence the last part of theorem 2.1.1 is a consequence of the following lemma which can be found in [3]. For the sake of completeness we give a proof.

Lemma 2.2.1. Let ρ and β be as above. The two integrals

$$\int_1^{\infty} \frac{\beta(x)}{x^2}\,dx \quad \text{and} \quad \int_0^{\alpha} \log^+ \rho(y)\,dy$$

converge or diverge together.

Proof. Without loss of generality we can assume that ρ is twice continuously differentiable with $\rho'' > 0$. By (2.1.4) and (2.1.6),

$$\alpha - \beta'(x) = y \quad \text{when} \quad x = \rho'(y).$$

By (2.1.5) $dx = \rho''(y)dy$.

Now

$$\int_{x_0}^{x_1} \frac{\beta(x)}{x^2}\,dx = [-\frac{\beta(x)}{x}]_{x_0}^{x_1} + \int_{x_0}^{x_1} \frac{\beta'(x)}{x}\,dx =$$

$$= O(1) + [\beta'(x)\log x]_{x_0}^{x_1} - \int_{x_0}^{x_1} \beta''(x)\log x\,dx .$$

Hence

$$\int_{x_0}^{x_1} \frac{\beta(x)}{x^2}\,dx = O(1) + (\alpha - y_1)\log \rho'(y_1) + \int_{y_0}^{y_1} \log \rho'(y)dy . \tag{2.2.2}$$

Suppose $\int_0^\alpha \log^+ \rho(y)dy < \infty$.

We use the following inequality, which we for the sake of simplicity give for the interval $[0,1[$.

If h is a positive, differentiable function and $h'(y) > 0$ for $y \in [0,1[$ then

$$\int_0^1 \log h'(y)dy \leqq 2 \int_0^1 \log h(y)dy - \log h(0) .$$

To see this we apply Jensen's inequality;

$$\int_0^1 (\log h'(y) - 2 \log h(y))dy = \int_0^1 \log(\frac{h'(y)}{h^2(y)})dy \leqq$$

$$\leqq \log \int_0^1 \frac{h'(y)dy}{h^2(y)} = \log[\frac{1}{h(0)} - \frac{1}{h(1)}] \leqq - \log h(0) .$$

Hence $\int_{y_0}^\alpha \log \rho'(y)dy < \infty$ for some $y_0 < \alpha$.

Since $\log \rho'$ is increasing this implies that $(\alpha-y)\log \rho'(y) \to 0$ as $y \to \alpha$.

Hence, by (2.2.2),

$$\int_1^\infty \frac{\beta(x)}{x^2} dx < \infty .$$

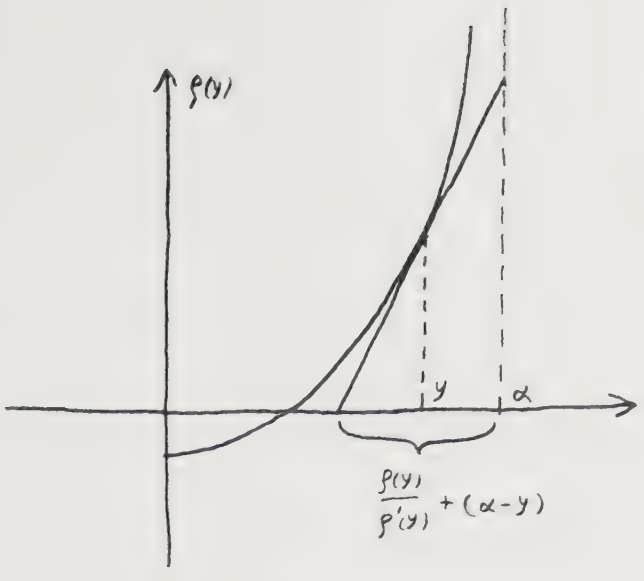
To handle the other case we use the identity

$$(2.2.3) \qquad \int_{y_0}^{y_1} \log \rho'(y)dy = \int_{y_0}^{y_1} \log(\frac{\rho'(y)}{\rho(y)})dy + \int_{y_0}^{y_1} \log \rho(y)dy .$$

Since ρ is convex we obtain

$$\frac{\rho(y)}{\rho'(y)} + (\alpha-y) \downarrow 0 \quad \text{when} \quad y \to \alpha . \quad \text{(See figure 3)}$$

Figure 3



In particular $\rho(y)/\rho'(y) \leqq y$.

Hence the first integral on the right hand side of (2.2.3) is bounded from below as y tends to α, so

$$\int_0^\alpha \log^+ \rho(y) = \infty \quad \text{implies} \quad \int_1^\infty \frac{\beta(x)}{x^2}\,dx = \infty .$$

2.3. Nonconvex θ.

One can ask if theorem 2.1.2 holds with less restrictive conditions on θ. If $\theta(t)/t \to \infty$ as $t \to \infty$ it does in general not hold. (cf section 3.4). We however conjecture that it holds if $\theta(t)/t \to \alpha \geqq 0$ as $t \to \infty$, $\int_1^\infty \theta(t)/t^2 dt = \infty$, and θ satisfies some regularity condition, e.g. θ even and $t\theta'(t)$ increasing for $t \geqq 0$. In the case $\alpha = 0$ this would lead to interesting results for sets of zeros of quasi-analytic functions (cf section 3.3).

The next theorem gives some information in this direction when $\alpha > 0$.

Theorem 2.3.1. Let $g \in L$, $g(t) = O(\frac{e^{-\theta(t)}}{1+t^2})$, where $\theta(t) = \alpha|t| + \beta(t)$ and β is even, nonnegative and $\beta(t) = o(t)$ as $t \to \infty$.

Let f be the Fourier transform of g and let $n(r)$ be the number of zeros of f in the interval $[0,r]$ counted according to their multiplicities. Suppose that there exists a convex function φ such that for some $C > 0$

$$\varphi(x) - C \leqq \log n(x) \leqq \varphi(x) + C$$

for $x \geqq x_0 > 0$.

If $\Lambda = \{\lambda_k\}_1^\infty$ is an increasing sequence of positive zeros of f and $f \not\equiv 0$, then

$$\sum_{k \leqq n} 1/\lambda_k^* = O\{-\log(\sum_{k \geqq n+1} \exp(-\frac{\pi}{2\alpha}|\lambda_k|))\}. \tag{2.3.1}$$

In the proof we use the following theorem by N. Levinson [13].

Theorem. Let $f \in L$, $f(x) = O(e^{-h(x)})$, $(x \to +\infty)$, where $h(x)$ is a positive, increasing, and twice differentiable function and $h''(x) \geqq c > 0$ for some c. Let the inverse function of $h'(x)$ be $H(x)$. Let g be the Fourier transform of f.

If

$$\lim_{x \to \infty} (\int_{-x}^{-1} + \int_1^x)(\frac{1}{t^2} - \frac{1}{x^2})\log|g(t)|dt + \pi H(x) = -\infty,$$

then f vanishes almost everywhere.

Proof of theorem 2.3.1. The convexity of φ implies that $\varphi(x)/x$ tends to a limit, s, as $x \to \infty$ $(0 \leqq s \leqq \infty)$. We first show that we can assume $s = \pi/2\alpha$.

The conditions on g imply that f is the restriction to R of a function, holomorphic and uniformly bounded in S_α. Since $f \not\equiv 0$ we clearly have $\Sigma \exp(-\frac{\pi}{2\alpha}|\lambda_k|) < \infty$. Hence

$$\overline{\lim_{k \to \infty}} \ (\tfrac{\pi}{2\alpha} \lambda_k/\log k) \geqq 1,$$

so

$$s = \lim_{x \to \infty} \frac{\varphi(x)}{x} = \underline{\lim_{k \to \infty}} \ (\frac{\log k}{\lambda_k}) \leqq \frac{\pi}{2\alpha} .$$

From $\beta(x) = o(x)$ it follows that for all $r < \frac{\pi}{2\alpha}$, $1/\lambda_k^* = O(e^{-r\lambda_k})$. Suppose $s < \frac{\pi}{2\alpha}$, then take s' such that $s < s' < \frac{\pi}{2\alpha}$. By the definition of s, $e^{-s'\lambda_k} = o(\frac{1}{k})$, hence $\Sigma \, 1/\lambda_k^* \leqq c_r \Sigma (1/k)^{r/s'}$ which is finite if $s' < r < \pi/2\alpha$. But then (2.3.1) is trivially satisfied.

We now estimate $|f(x)|$. By "Schwarz's lemma for the strip S_α",

$$|f(x)| \leqq M \prod_k \left| \frac{\exp(\pi x/2\alpha) - \exp(\pi\lambda_k/2\alpha)}{\exp(\pi x/2\alpha) + \exp(\pi\lambda_k/2\alpha)} \right| . \tag{2.3.2}$$

Taking logarithms and using the inequality $\log(\frac{1-t}{1+t}) \leqq -2t$ we obtain

$$\log |f(x)| \leqq -2 \sum_{k=1}^{\infty} \exp(-\tfrac{\pi}{2\alpha}|x-\lambda_k|) + O(1) \leqq \tag{2.3.3}$$

$$\leqq -2 \exp(\tfrac{\pi x}{2\alpha}) \cdot \int_x^\infty \exp(-\tfrac{\pi r}{2\alpha}) dn(r) + O(1) .$$

Integration by parts gives

$$\exp(\tfrac{\pi x}{2\alpha}) \int_x^\infty e^{-\frac{\pi r}{2\alpha}} dn(r) = \tag{2.3.4}$$

$$= \exp(\tfrac{\pi x}{2\alpha}) [\exp(-\tfrac{\pi r}{2\alpha}) n(r)]_x^\infty + \tfrac{\pi}{2\alpha} \exp(\tfrac{\pi x}{2\alpha}) \int_x^\infty \exp(-\tfrac{\pi r}{2\alpha}) n(r) dr .$$

By our assumptions on φ, for $x \geq 0$,

$$\varphi(x) = \frac{\pi}{2\alpha} x - \gamma(x)$$

where γ is concave and $\gamma'(x) \to 0$ as $x \to \infty$. We know that the integral on the left side of (2.3.4) converges. Hence the integral on the right side also converges, that is

$$\int_x^\infty \exp(-\gamma(r))dr < \infty .$$

We conclude that $\gamma(x) \to \infty$ as $x \to \infty$.

Put

$$h_0(x) = \exp(\frac{\pi x}{2\alpha}) \int_x^\infty \exp(-\gamma(r))dr .$$

By (2.3.3) and (2.3.4) we then have

$$\log |f(x)| \leq -\frac{\pi}{\alpha} h_0(x) + n(x) + O(1) \leq -c_1 h_0(x) = -h(x)$$

for some $c_1 > 0$, since by l'Hospital's rule

$$\lim_{x \to \infty} (\frac{\exp(\varphi(x))}{h_0(x)}) = \lim_{x \to \infty} \frac{\exp(-\gamma(x))}{\int_x^\infty \exp(-\gamma(t))dt} = 0 .$$

Since $h_0'(x) = \frac{\pi}{2\alpha} h_0(x) - \exp(\varphi(x))$ we have

(2.3.5) $$h'(x)/h(x) \to \frac{\pi}{2\alpha} \quad \text{as} \quad x \to \infty .$$

To see that $h''(x) \geq c > 0$ for x sufficiently large, just note that $h_0''(x) = (\frac{\pi}{2\alpha})^2 h_0(x) - \frac{\pi}{2\alpha} \exp(\varphi(x)) - \varphi'(x)\exp(\varphi(x))$. Hence $h''(x)/h(x) \to (\frac{\pi}{2\alpha})^2$.

We now put the pieces together. Put $a_k = \exp(-\frac{\pi\lambda_k}{2\alpha})$ and $s_n = \sum_{k \geq n} a_k$. It is readily seen that

$$(2.3.6) \qquad (1-\frac{a_1}{s_1})(1-\frac{a_2}{s_2}) \ \dots \ (1-\frac{a_n}{s_n}) = \frac{s_{n+1}}{s_1} \ .$$

Suppose (2.3.1) does not hold. Then, for a sequence of n:s

$$- \log(s_{n+1}) = o\{ \sum_{k \leqq n} 1/\lambda_k^* \} \ .$$

From (2.3.6) we then see that there is a sequence of n:s such that

$$-\log(1-\frac{a_n}{s_n}) = o(1/\lambda_n^*) \ .$$

Hence $a_n/s_n = o(1/\lambda_n^*)$ and consequently

$$\lambda_n^* = o(\exp(\frac{\pi}{2\alpha} \lambda_n) \sum_{k \geqq n} \exp(- \frac{\pi}{2\alpha} \lambda_k)) =$$

$$= o(-n(\lambda_n) + \frac{\pi}{2\alpha} h_0(\lambda_n)) = o(h(\lambda_n)) = o(h'(\lambda_n)).$$

Hence

$$\overline{\lim_{n \to \infty}} \ \frac{2}{\pi} \int_{\lambda_n^*}^{h'(\lambda_n)} \frac{\theta(t)}{t^2} \, dt = \infty \ ,$$

that is

$$\overline{\lim_{n \to \infty}} \ \{ \int_1^{h'(\lambda_n)} \frac{\theta(t)}{t^2} - \frac{\pi}{2} \lambda_n \} = \infty \ .$$

By the definition of H,

$$\overline{\lim_{x \to \infty}} \ \{ \int_1^x \frac{\theta(t)}{t^2} \, dt - \frac{\pi}{2} H(x) \} = \infty \ .$$

Since $\beta(t) = o(t)$,

$$\lim_{x \to \infty} \left(\int_{-x}^{-1} + \int_{1}^{x} \right) \left(\frac{1}{t^2} - \frac{1}{x^2} \right) \log |g(t)| dt + \pi H(x) = -\infty ,$$

which by Levinsons theorem yields $f \equiv 0$, a contradiction. Hence (2.3.1) holds and the theorem is proved. (Actually g is the inverse Fourier transform of f but this does not affect the conclusion).

Example. With $\theta(x) = \frac{\pi}{2} |x| + \frac{|x|}{\log(|x| + 1)}$ we get $\lambda^* \approx e^{\lambda}/\lambda$ hence $f \not\equiv 0$ implies $\sum_{k \leq n} \lambda_k / e^{\lambda_k} = O\{-\log(\sum_{k \geq n+1} e^{-\lambda_k})\}$. In the case $\lambda_k = \log k + a \log\log k$ this means that $a \geq 2$.

2.4. A converse theorem.

It is natural to ask if theorem 2.1.1 is best possible in some sense, i.e. given that the series $\Sigma 1/\lambda_n^*$ converges, can we find $f \not\equiv 0$ with Λ as zero set and satisfying the growth condition? We give a partial answer to this question.

In this section Λ is a doubly infinite sequence of real numbers, $\{\lambda_k\}_{-\infty}^{\infty}$, such that $\lambda_k \leq \lambda_{k+1}$ and $\lambda_0 = 0$.

Theorem 2.4.1. Let ρ be even and convex for $|y| < \alpha$, suppose $\rho(0) < 0$ and $\rho(y) \to \infty$ as $y \to \alpha$. Assume that ρ^*, the convex conjugate of ρ, has an analytic extension with a uniformly bounded derivative in the region $|z| > R$, $|\arg(z)| < \delta$, for some $R > 0$, $\delta > 0$. Let λ_k^* be defined as in theorem 2.1.1.

If $\Sigma \frac{1}{\lambda_k^*} < \infty$ there exists a function $f \not\equiv 0$, holomorphic in the strip S_α, such that

$$|f(x+iy)| \leq \exp(\rho(y)) ,$$

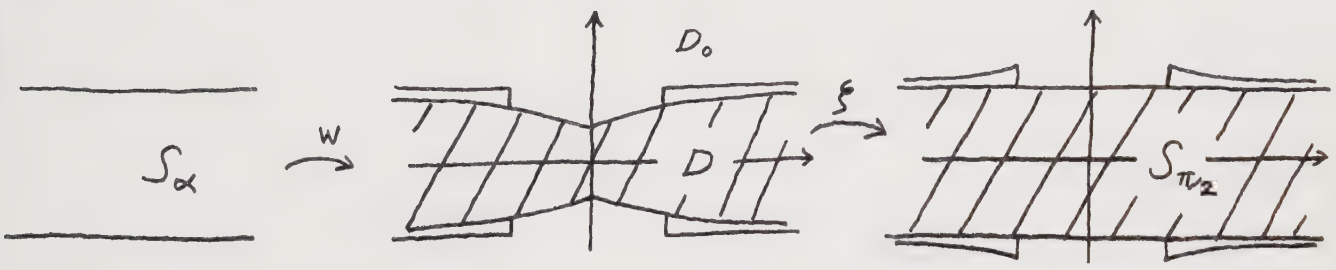
and with Λ as its set of zeros (multiplicities counted).

Proof. Put $\rho^* = \theta$ and let θ also denote the extension of ρ^*. Let D and ψ be defined as in the proof of theorem 2.1.1 (2.1.9 ff). Let ζ be the conformal mapping of D onto $S_{\pi/2}$, such that $\zeta(0) = 0$, $\zeta'(0) > 0$. By (2.1.19)

$$|\zeta(\lambda_k)| = \log(\lambda_k^*) + O(1) . \tag{2.4.1}$$

Before constructing f we take care of some technicalities.

Figure 4



We claim that:

(*) it is possible to continue ζ analytically to a region $D_0 = D \cup \{w = u+iv: |u| > x_0, |v| < \alpha\}$, (see figure 4) for some $x_0 > 0$ in such a way that ζ' is uniformly bounded for $z \in D_0, |\mathrm{Re}\,(z)| > x_0$. ($|\zeta'| \leq M$).

We postpone the proof of (*), but observe that one consequence is that we can assume that x_0 has been chosen so that the image of D_0 under

ζ is contained in a strip strictly contained in S_π; let us take $S_{3\pi/4}$ to be this strip.

Let $w(z)$ map S_α conformally onto D_0 with $w(0) = 0$, $w'(0) > 0$, so that symmetry with respect to the axes is preserved.

For later use we note that the argument leading to (2.1.17) applies to w, hence

$$|v| = |\mathrm{Im}\, w(x+iy)| \leq |y| . \tag{2.4.2}$$

It is sufficient to find f with $f(\lambda_k) = 0$ for $|\lambda_k| > N$ where N is some fixed positive number, since

$$P_n(z) = \prod_{|\lambda_k| \leq N} \left(\frac{\exp(\pi z/2\alpha) - \exp(\pi\lambda_k/2\alpha)}{\exp(\pi z/2\alpha) + \exp(\pi\lambda_k/2\alpha)}\right)$$

has modulus less than 1 in S_α and $P_N f$ will have the prescribed properties.

Put

$$\Lambda(N) = \{k: |\lambda_k| > N\},$$

$$\sigma(z) = \zeta(w(z)),$$

and

$$F_N(z) = \prod_{k \in \Lambda(N)} \left(\frac{\exp(\sigma(z)) - \exp(\sigma(\lambda_k))}{\exp(\sigma(z)) + \exp(\sigma(\lambda_k))}\right) \quad \text{for} \quad z \in S_\alpha .$$

By the distortion theorem and (2.4.1),

$$|\sigma(\lambda_k)| = |\zeta(w(\lambda_k))| = |\zeta(\lambda_k + 0(1))| = |\zeta(\lambda_k)| + 0(1) =$$

$$= \log(\lambda_k^*) + 0(1) .$$

Thus $\Sigma \exp(-|\sigma(\lambda_k)|) < \infty$, hence the product defining F_N is convergent if $\sigma(z) \in S_\pi$ and a fortiori for $z \in S_\alpha$.

We will need a rough estimate for

$$K(r,\theta) = \log \left|\frac{1 - re^{i\theta}}{1 + re^{i\theta}}\right| \quad \text{when} \quad 0 \leqq r \leqq 1, \ |\theta| \leqq \frac{3\pi}{4} .$$

We have

$$K(r,\theta) = \frac{1}{2} \log \left|\frac{1 + r^2 - 2r \cos \theta}{1 + r^2 + 2r \cos \theta}\right| =$$

$$= \frac{1}{2} \log \left|1 - \frac{4r \cos \theta}{1 + r^2 + 2r \cos \theta}\right| \leqq -4r \cos \theta .$$

Hence

$$K(r,\theta) \leqq \begin{cases} 4r(|\theta| - \frac{\pi}{2}) & \text{if} \quad \frac{\pi}{2} \leqq |\theta| \leqq \frac{3\pi}{4} \\ 0 & \text{if} \quad |\theta| \leqq \frac{\pi}{2} \end{cases} . \tag{2.4.3}$$

For a fixed $z \in S_\alpha$ we put

$$\begin{cases} r_k = \exp(- \ |\mathrm{Re}\ \sigma(z) - \sigma(\lambda_k)|) , \\ \theta = \mathrm{Im}\ \sigma(z) . \end{cases}$$

By the definition of F_N and (2.4.3) we have

$$\log |F_N(z)| = \sum_{k \in \Lambda(N)} \log \left|\frac{1 - r_k e^{i\theta}}{1 + r_k e^{i\theta}}\right| \leqq \tag{2.4.4}$$

$$\leqq \begin{cases} 4(\theta - \frac{\pi}{2}) \sum r_k & \text{if} \quad \frac{\pi}{2} \leqq \theta \leqq \frac{3\pi}{4} , \\ 0 & \text{if} \quad 0 \leqq \theta \leqq \frac{\pi}{2} . \end{cases}$$

By reasons of symmetry it is sufficient to consider $z = x+iy$, $x \geqq 0$, $y \geqq 0$.

Now $\theta = \mathrm{Im}\ \zeta(u+iv)$ where $u+iv = w(x+iy)$. By (2.4.2), $v \leqq y$.

We now show that

$$(2.4.5) \qquad \theta - \frac{\pi}{2} \leqq \begin{cases} M(y - \psi(u)) & \text{if } \psi(u) \leqq y , \\ 0 & \text{if } \psi(u) > y . \end{cases}$$

If $\psi(u) \leqq v$, then since $|\zeta'| \leqq M$,

$$\theta - \frac{\pi}{2} = \operatorname{Im} \zeta(u+iv) - \operatorname{Im} \zeta(u + i\psi(u)) \leqq$$

$$\leqq M|v - \psi(u)| = M(v - \psi(u)) \leqq M(y - \psi(u)) .$$

If $\psi(u) > v$, then by the construction $\theta < \frac{\pi}{2}$, hence the rest of (2.4.5) is trivially satisfied.

The distortion theorem applied to ζ yields

$$|\operatorname{Re} \zeta(u+iv)| - \frac{\pi}{2} \int_0^u \frac{dt}{\psi(t)} = O(1), \ w \in D_0 .$$

Actually it follows only for $v < \psi(u)$, but since $|\zeta'| \leqq M$ in D_0 it holds for all $w \in D_0$.

As in (2.1.11) we put $\varphi(u) = \frac{\pi}{2} \int_0^u \frac{dt}{\psi(t)}$. Hence

$$|\operatorname{Re} \sigma(z) - \sigma(\lambda_k)| = |\operatorname{Re} \zeta(u+iv) - \zeta(\lambda_k)| + O(1) =$$

$$= |\varphi(u) - \varphi(\lambda_k)| + O(1) .$$

Substitution in (2.4.4) gives

$$(2.4.6) \qquad \log |F_N(z)| \leqq \begin{cases} 4MC_0(y - \psi(u)) \sum_{k \in \Lambda(N)} \exp\{- |\varphi(u) - \varphi(\lambda_k)|\} \\ 0 \end{cases}$$

for $y \geqq \psi(u)$ and $y < \psi(u)$ respectively. Note that M and C_0 are independent of y, u, and Λ.

We shall prove that N can be chosen so that

$$(2.4.7) \qquad \sum_{k \in \Lambda(N)} \exp\{- |\varphi(u) - \varphi(\lambda_k)|\} \leqq 3e^{\varphi(u)} .$$

Recall that $\lambda_0 = 0$ and $u \geqq 0$. We split the sum in three parts,

$$e^{-\varphi(u)} \sum_{\lambda_k < 0} e^{\varphi(\lambda_k)} + e^{-\varphi(u)} \sum_{0 \leqq \lambda_k < u} e^{\varphi(\lambda_k)} + e^{\varphi(u)} \sum_{\lambda_k \geqq u} e^{-\varphi(\lambda_k)}.$$

All summation is taken over $k \in \Lambda(N)$. The series $\sum_{\lambda_k \geqq 0} e^{-\varphi(\lambda_k)}$ is convergent and has decreasing terms. This implies the existence of a number k_0 such that $\exp(-\varphi(\lambda_k)) < \frac{1}{k}$ if $k \geqq k_0$. Consequently, if $N > \lambda_{k_0}$, and $k \in \Lambda(N)$, then $k < \exp(\varphi(\lambda_k))$, hence $\lambda_k < u$ implies $k < \exp(\varphi(u))$. This shows that the middle sum has less than $\exp(\varphi(u))$ terms, hence the contribution from the middle term will not exceed $\exp(\varphi(u))$.

Since $\Sigma \exp(-\varphi(|\lambda_k|))$ converges and $\exp(-\varphi(u)) \leqq \exp(\varphi(u))$ it is clear that if N is large enough (2.4.7) will hold.

By (2.4.6),

$$\log |F_N(z)| \leqq 12MC_0(y - \psi(u))e^{\varphi(u)}, \quad (\psi(u) \leqq y).$$

Putting $u = \frac{2}{\pi} \int_0^s \theta'(e^t)dt$ we have, by (2.1.12), $\varphi(u) = s$ and by the construction of D, $\psi(u) = \theta'(e^s)$.

Hence

$$\log |F_N(z)| \leqq 12MC_0 \max_{\substack{\theta'(e^s) \leqq y \\ s \geqq 0}} (y - \theta'(e^s))e^s \leqq$$

$$\leqq 12MC_0 \sup_{t \geqq 0} (yt - \theta(t) + \theta(t) - t\theta'(t)) \leqq$$

$$\leqq 12MC_0 \sup_{t \geqq 0} (yt - \theta(t)) + C_1 = 12MC_0\theta^*(y) + C_1.$$

Since $\theta(t) - t\theta'(t)$ is decreasing.

We now replace the original sequence Λ with Λ' where each λ_k is repeated $1+[12MC_0]$ times. Since M and C_0 are independent of Λ we can in the same way as before find N and F_N such that

$$\log |F_N(z)| \leqq 12MC_0\theta^*(y) + C_1 .$$

But $F_N(z) = f_N(z)^{1+[12MC_0]}$ where f_N has precisely the zeros $\{\lambda_k\}$. Hence

$$|f_N(z)| \leqq C \exp(\theta^*(y)) .$$

According to our remark earlier this completes the proof of the theorem except for the statement (*).

To prove it we start by defining

$$(2.4.8) \qquad g(z) = \frac{2}{\pi} \int_0^z \theta'(e^u)du + i\theta'(e^z) .$$

The path of integration follows the real line to some $u_0 > \log R$ and thereafter it lies in the region where $\theta(e^z)$ is analytic, $z = x+iy$, $x > \log R$, $|y| < \delta$. By hypothesis $|\theta'(e^z)| \leqq M_0$ in that region.

The half line $]\log R,\infty[$ is mapped by g onto the upper right boundary curve of D except for a segment near the imaginary axis.

Since $\theta'(e^x) \to \alpha$ as $x \to \infty$ and θ' is bounded it follows from Montel's theorem that $\theta'(e^z) \to \alpha$ uniformly for $|y| \leqq \delta_0 < \delta$ when $x \to \infty$. It easily follows that $\frac{d}{dz}(\theta'(e^z)) \to 0$ uniformly for $|y| \leqq \delta_1 < \delta$ when $x \to \infty$.

Fix $\delta' < \delta$. Differentiating (2.4.8) yields

$$g'(z) = \frac{2}{\pi} \theta'(e^z) + i \frac{d}{dz} \theta'(e^z) .$$

Hence, given $\varepsilon > 0$ we can find K_ε such that

$$(2.4.9) \qquad |g'(z) - \frac{2\alpha}{\pi}| < \varepsilon \quad \text{for} \quad x > K_\varepsilon, \ |y| \leqq \delta' .$$

Choosing ε small enough makes g a conformal mapping defined on

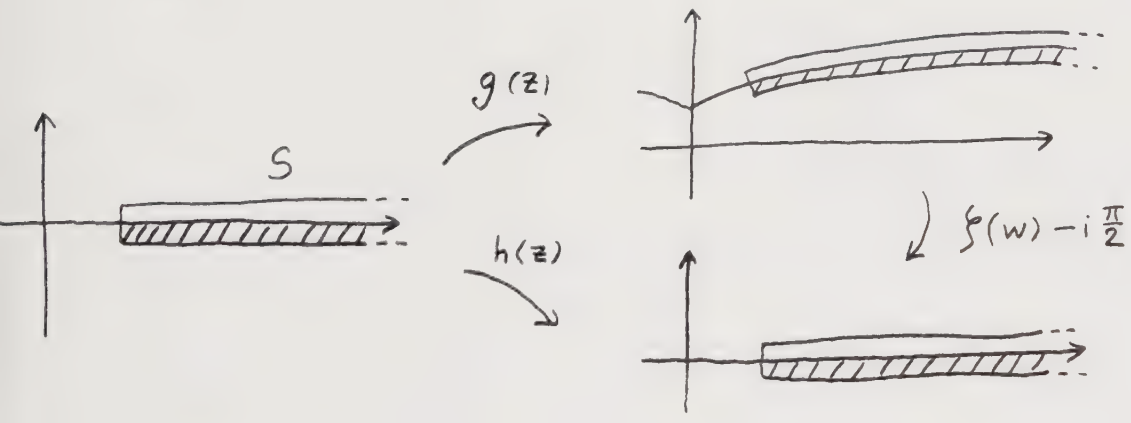
$$S = \{z = x+iy: \ x > K_\varepsilon, \ |y| \leqq \delta'\} .$$

It is clear that we have the correspondence

$$\begin{cases} g(z) \in \bar{D} & \text{if} \quad \mathrm{Im}(z) \leqq 0 \\ g(z) \in D & \text{if} \quad \mathrm{Im}(z) > 0 \end{cases}$$

when $z \in S$ and δ' is small enough.

Figure 5



Put

$$h(z) = \zeta(g(z)) - i\,\frac{\pi}{2} , \quad \text{for} \quad z \in S, \ \mathrm{Im}(z) \leqq 0,$$

(ζ has a continuous extension to $\bar{D}$). Since h is continuous and takes real values for z real, we can extend it by Schwartz's reflection principle to all of S. We denote the extended function by $\tilde{h}$.

By (2.4.9) it is clear that there is $r > 0$ such that the image of S under g contains a disc of radius r around each point of D's contour in the first quadrant (except for a segment near the imaginary axis). Hence we can find x_0 such that

$$D_1 = D \cup \{w = u+iv:\ u \geqq x_0 - r/2,\ 0 \leqq v \leqq \frac{\pi}{2} + \frac{r}{2}\}$$

is contained in $D \cup g(S)$.

For $w \in g(S)$ we define $\tilde{\zeta}(w) = \tilde{h}(g^{-1}(w)) + i\frac{\pi}{2}$, and for $w \in D$, $\tilde{\zeta}(w) = \zeta(w)$. The definition of $\tilde{h}$ shows that $\tilde{\zeta}$ is well defined in $D \cup g(S)$. Finally $\tilde{\zeta}$ is extended symmetrically to

$$D \cup \{w = u+iv:\ |u| \geqq x_0 - \frac{r}{2},\ |v| \leqq \frac{\pi}{2} + \frac{r}{2}\} = D_2.$$

It remains to prove that $\tilde{\zeta}$ has uniformly bounded derivative in the region $D_0 = \{w = x+iy:\ |u| \geqq x_0,\ |v| \leqq \frac{\pi}{2}\}$. By the construction, $|\mathrm{Im}(\tilde{\zeta}(w))| < \pi$ for $w \in D_2$. Put $H(w) = \exp(i\tilde{\zeta}(w))$. For $w \in D_2$, $e^{-\pi} < |H(w)| < e^{\pi}$, and it easily follows that H' is uniformly bounded in D_0. But $|H'(w)| = |\tilde{\zeta}'(w)| \cdot |H(w)|$ which proves the boundedness. Thus the proof of (*) is complete.

3. Approximation by translates.

We use the results of ch. 2 to prove some theorems concerning approximation in the spaces $L(R)$, $L^2(R)$, and $C_0(R)$ by linear combinations of translates of a given function satisfying certain regularity properties expressed by conditions on its Fourier transform.

Let Λ be a set of real numbers. Given a function f, we define $\Lambda(f)$ to be the set of finite linear combinations of the functions $f_\lambda = f(\cdot - \lambda)$, $\lambda \in \Lambda$.

Let A be the class of even, positive, convex functions θ, such that $\theta(x)/x \to \alpha \neq 0, \infty$, as $x \to \infty$.

Further we let B be the class of $\theta \in A$, such that θ has a holomorphic extension with uniformly bounded derivative to the region $|z| > R$, $|\arg(z)| < \delta$, for some $R > 0$, $\delta > 0$ (cf thm 2.4.1).

Given $\lambda \in R$ we define λ^* by (2.1.20). Which function θ should be used will be clear from the context. If two such functions are involved θ and θ_1, then λ^* is defined with respect to θ and λ^+ with respect to θ_1.

We let $f * g$ denote the convolution of f and g,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-t)g(t)dt.$$

Given a function f, $\check{f}$ is defined by $\check{f}(x) = f(-x)$.

3.1 Müntz - Szász type theorems

In this section we prove generalizations of the Müntz - Szász theorem viewed as a theorem on approximation by translates in the way we showed in the introduction.

In this section Λ will be a countable set $\{\lambda_k\}$, i.e. all λ_k are different, except in the first theorem, where we allow multiple λ_k.

In this case $\Lambda(f)$ consists of all finite linear combinations of the functions $f^{(n)}(\cdot - \lambda_k)$, $0 \leq n \leq m_k - 1$ (m_k the multiplicity of λ_k).

Theorem 3.1.1 Let $\theta \in A$, let $f \in L^2$ and suppose $\hat{f}(t) = O(e^{-\theta(t)})$, $(|t| \to \infty)$, and $\hat{f}(t) \neq 0$ a.e.

If $\Sigma \frac{1}{\lambda_k^*} = \infty$, then $\Lambda(f)$ is dense in L^2.

Proof: Take $g \in L^2$ such that $g \perp \Lambda(f)$.

Put $$F(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\lambda)g(x)dx.$$

By hypothesis $F(\lambda_k) = 0$.

Since $F = \overset{\vee}{f} * g$, we have after taking Fourier transforms

$$\hat{F} = \hat{f}^{\vee} \cdot \hat{g}. \tag{3.1.1}$$

For all $n \geq 0$, $t^n \hat{f}(t) \in L$. By inverting the relation

$$(it)^n \hat{F}(t) = (it)^n \hat{f}^{\vee}(t)\hat{g}(t), \text{ (a.e.)}$$

we have $F^{(n)} = D^n(\overset{\vee}{f}) * g$.

Since $g \perp f^{(n)}_{\lambda_k}$ for $n = 0, \ldots, m_k - 1$, it follows that

$$F^{(n)}(\lambda_k) = 0, \quad n = 0, \ldots, m_k - 1.$$

Hence F has a zero of at least the multiplicity n_k at the point λ_k.

By (3.1.1) $e^{yt}\hat{F}(t) \in L$, when $|y| < \alpha$.

Hence F can be analytically extended to S_α by

$$F(x+iy) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} e^{it(x+iy)}\hat{F}(t)dt.$$

Put $\theta_1(t)$ = the largest convex minorant to $\theta(t) - \log(1+t^2) + C$ where C is chosen so that $\theta_1 > 0$.

Since $|\hat{F}(t)| \leq C_1 \exp(-\theta(t)) \frac{\hat{g}(t)}{1+t^2}$, we find as in (2.1.3),

$$|F(x+iy)| \leq C_2 \exp(\theta_1^*(y)).$$

By hypothesis $\Sigma \frac{1}{\lambda_k^*} = \infty$ and proceeding as in theorem 2.1.2 we obtain $\Sigma \frac{1}{\lambda_k^+} = \infty$.

By theorem 2.1.2 it follows that $F \equiv 0$. Since $\hat{f} \neq 0$ a.e., $\hat{g} = 0$ a.e. Hence $g = 0$ a.e., which proves that $\Lambda(f)$ is dense in L^2.

Remark: Specializing to the case $\hat{f}(t) = O(t^n \cdot e^{-\alpha|t|})$, which we encountered in our discussion of the Müntz approximation theorem, we find that a sufficient condition for $\Lambda(f)$ to span L^2 is $\Sigma \exp(-\frac{\pi}{2\alpha}|\lambda_k|) = \infty$.

Theorem 3.1.2 Let $\theta \in B$, $f \in L^2$ and suppose

$$(1/\hat{f}(t))\, e^{-\theta(t)} \in L^2.$$

If $\Sigma \frac{1}{\lambda_k^*} < \infty$ then $\Lambda(f)$ is not dense in L^2.

Proof By theorem 2.4.1 we can find $g \not\equiv 0$, holomorphic in S_α, such that $g(\lambda_k) = 0$ and

$$|g(x+iy)| \leq \exp(\theta^*(y)).$$

Put

$$g_0(z) = g(z)/(z^2 + 2\alpha^2).$$

In the integral

$$\hat{g}_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g_0(x) e^{-itx} dx$$

we can change the path of integration to the line $\mathrm{Im}(z) = -y$ without changing its value.

Hence $|\hat{g}_0(t)| \leq C\, e^{-ty} \exp(\theta^*(y))$ and minimizing over y we obtain

$$|\hat{g}_0(t)| \leq C \exp(-\theta^{**}(t)) = C \exp(-\theta(t)). \tag{3.1.2}$$

Define φ by $\hat{g}_0(t) = \hat{f}^\vee(t)\varphi(t)$. By (3.1.2) and by hypothesis, $\varphi \in L^2$. Since

$$g_0(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\lambda)\hat{\varphi}^\vee(x)dx,$$

we conclude that $\hat{\varphi}^{\vee}$ is a nontrivial element in L^2 orthogonal to $\Lambda(f)$, which proves the theorem.

In the special case $f(t) = O(e^{-\alpha|t|})$, the preceding theorem does not give much information. We cover this case in the next theorem.

Theorem 3.1.3 Let β be an even, positive function, concave for $t \geq 0$, and suppose

$$\int_1^\infty \frac{\beta(t)}{t^2}\, dt < \infty .$$

Suppose $f \in L^2$, $|\hat{f}(t)| \geq C \exp(-\alpha|t| - \beta(t))$, $\alpha > 0$.
If $\Sigma \exp(-\frac{\pi}{2\alpha}|\lambda_k|) < \infty$, then $\Lambda(f)$ is not dense in L^2.

Proof: Put $\beta_0(t) = \beta(t) + \log(1+|t|)$ and $p(x) = e^x/\beta_0(e^x)$ for $x \geq 0$. Since β_0 is positive and concave for $t \geq 0$, $p(x)$ must be increasing. Put $a = \alpha + 1/p(0)$.
If $z \in \overline{S}_a$

$$|H(z)| = \left|\exp\left(-\left\{\exp\left(\frac{\pi z}{4a}\right) + \exp\left(-\frac{\pi z}{4a}\right)\right\}\right)\right| \leq \exp\left(-\frac{1}{\sqrt{2}}\exp\left(\frac{\pi|x|}{4a}\right)\right) = \exp(-c\exp(bx)) .$$

Put $p_0(x) = p(bx)$ and $\psi(x) = \alpha + \frac{1}{p_0(x)}$. Let D be the region $\{z = x+iy : |y| < \psi(|x|)\}$. Map D conformally onto S_α by ϕ such that $\phi(0) = 0$, $\phi'(0) > 0$. Put $\mu_k = \phi(\lambda_k)$. By the distortion theorem

$$\mu_k = \int_0^{\lambda_k} \frac{dx}{\psi(x)} + O(1) = \lambda_k + \int_0^{\lambda_k} \frac{dx}{1+(p_0(x)/\alpha)} + O(1).$$

Since the last integral is less than

$$\alpha \int_0^\infty \frac{dx}{p_0(x)} = \frac{\alpha}{b}\int_0^\infty \frac{dy}{p(y)} = \frac{\alpha}{b}\int_0^\infty \frac{\beta_0(e^y)}{e^y}\, dy = \frac{\alpha}{b}\int_0^\infty \frac{\beta_0(t)}{t^2}\, dt < \infty ,$$

we see that

$$\Sigma \exp(-\tfrac{\pi}{2\alpha}|\mu_k|) < C_1 \Sigma \exp(-\tfrac{\pi}{2\alpha}|\lambda_k|) < \infty .$$

Hence we can find $G \not\equiv 0$, holomorphic in S_α, with

$G(\mu_k) = 0$ and $|G| < 1$.

Put $F(z) = G(\phi(z)) \frac{H(z)}{z^2+2a^2}$, $z \in D$.

Its Fourier transform

$$\hat{F}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itx} F(x)dx$$

can equally well be found by integrating along the lower boundary of D.

Suppose $t \geq 0$. Substitute $\zeta = x - i\alpha - i \frac{1}{p_0(|x|)}$ in the integral.

Since

$$d\zeta = (1 + i \frac{p_0'(|x|)}{p_0(|x|)^2})dx$$

and

$$\frac{p_0'(x)}{p_0(x)} = b(1 - \frac{\beta_0'(e^{bx})e^{bx}}{\beta_0(e^{bx})}) \leq b$$

and p_0 is increasing, there is $C_2 > 0$ such that $|d\zeta| \leq C\,dx$.

Hence $|\hat{F}(t)| \leq C_3 \int_0^{\infty} \exp(-\alpha t - \frac{t}{p_0(x)} - c\,e^{bx}) \frac{dx}{x^2+a^2}$.

Choose τ such that $c\,e^{b\tau} = t/p_0(\tau)$. This is possible for all $t \geq M$ for some $M > 0$. If $0 \leq x \leq \tau$, then

$$\frac{t}{p_0(x)} + c\,e^{bx} \geq \frac{t}{p_0(x)} \geq \frac{t}{p_0(\tau)},$$

and if $x > \tau$, then

$$\frac{t}{p_0(x)} + c\,e^{bx} \geq c\,e^{b\tau} = \frac{t}{p_0(\tau)}$$

Since $p_0(\tau) \to \infty$ when $t \to \infty$ there is $M_1 > M$ such that $t \geq M_1$ implies $p_0(\tau) > \frac{1}{c}$ and by the definition of τ, $\tau < \frac{1}{b} \log t$.

Hence

$$\frac{t}{p_0(\tau)} > \frac{t}{p_0(\frac{1}{b} \log t)} = \frac{t}{p(t)} = \beta_0(t),$$

and

$$|\hat{F}(t)| \leq C_4 \exp(-\alpha t - \beta_0(t)) \quad \text{for} \quad t \geq M_1.$$

By the corresponding estimates for $t \leq 0$ and appropriate choice of C we obtain

$$|\hat{F}(t)| \leq C \exp(-\alpha|t| - \beta(t)) = C \frac{\exp(-\alpha|t| - \beta(t))}{(1 + |t|)} .$$

Then, proceeding in the same way as in the proof of theorem 3.1.2 we write $\hat{F} = \hat{f}^{\vee} \cdot \varphi$, $\varphi \in L^2$ and $\hat{\varphi}^{\vee}$ will be a nontrivial element of L^2 orthogonal to $\Lambda(f)$. The theorem is proved.

We proceed to study approximation in L.

Theorem 3.1.4 Let $\theta \in A$, $f \in L$, suppose that $\hat{f}$ is differentiable with $D\hat{f}(t) = O(e^{-\theta(t)})$ and that $\hat{f}(t) \neq 0$, $\forall t$.
If $\Sigma \frac{1}{\lambda_k^*} = \infty$ then $\Lambda(f)$ is dense in L.

Proof: First we observe that $\hat{f}(t) = O(e^{-\theta(t)})$: Since $\theta'(t) > \frac{\alpha}{2}$ for t sufficiently large,

$$|\hat{f}(x)| \leq \int_{|x|}^{\infty} e^{-\theta(t)} dt \leq \frac{2}{\alpha} \int_{|x|}^{\infty} e^{-\theta(t)} \theta'(t) dt = \frac{2}{\alpha} e^{-\theta(x)}.$$

For almost all x,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixt} \hat{f}(t) dt. \tag{3.1.3}$$

We can assume that f is everywhere given by (3.1.3), since this change does not affect the statement of the theorem.

Since $\hat{f}(t)e^{yt} \in L$ for $|y| < \alpha$, it is possible to extend f analytically to S_α. Put $f_y(x) = f(x+iy)$. By (3.1.3) and the inversion theorem for the Fourier transform,

$$\hat{f}_y(t) = e^{-yt} \hat{f}(t) \quad \text{a.e.} . \tag{3.1.4}$$

By continuity (3.1.4) holds for all t.

Letting $\theta_1(t)$ be the largest convex minorant to $\theta(t) - \log(1+t^2) + C_0$ for suitable C_0, we obtain

$$\| \hat{f}_y \|_2^2 \leq C \int_0^{\infty} (\exp(|y|t - \theta_1(t)))^2 \frac{dt}{1+t^2} \leq C \exp(2\theta_1^*(y)) .$$

Hence

(3.1.5) $$\| \hat{f}_y \|_2 \leq C_1 \exp(\theta_1^*(y)) .$$

We also wish to estimate $\| Df_y \|_2$. Differentiating (3.1.4) yields $D\hat{f}_y = e^{-yt} D\hat{f} - ye^{-yt}\hat{f}$ and both terms can be handled as above. Hence

$$\| D\hat{f}_y \|_2 \leq C_2 \exp(\theta_1^*(y)) .$$

Carlson's inequality gives

(3.1.6) $$\| f_y \|_1 \leq C_3 \exp(\theta_1^*(y)) .$$

Take $\varphi \in L^\infty$, $\varphi \perp \Lambda(f)$ and put $F = \check{f} * \varphi$. We claim that F is holomorphic in S_α . Then, since $|F(x+iy)| \leq C_3 \| \varphi \|_\infty \exp(\theta_1^*(y))$, it follows as in the proof of theorem 3.1.1 that $F \equiv 0$, hence $\varphi = 0$ a.e. by Wieners approximation theorem.

To prove analyticity of F it is by Morera's and Fubini's theorems sufficient to prove continuity.

For $z, z_0 \in S_\alpha$, let $g_z(x) = f(x-z) - f(x-z_0)$.

Then $\hat{g}_z(t) = \hat{f}(t)(e^{-izt} - e^{-iz_0 t})$. When $z \to z_0$ both $\| \hat{g}_z \|_2$ and $\| D\hat{g}_z \|_2$ tend to zero, by dominated convergence, and we use Carlson's inequality again to conclude $\| g_z \|_1 \to 0$, hence $F(z) \to F(z_0)$ and the proof is complete.

<u>Theorem 3.1.5</u>. <u>Suppose</u> $\theta \in A$, $f \in L$ <u>and</u>

$$(1/\hat{f}(t))e^{-\theta(t)} \in L .$$

<u>If</u> $\sum \frac{1}{\lambda_k^*} < \infty$ <u>then</u> $\Lambda(f)$ <u>is not dense in</u> L.

Proof: We can use the proof of theorem 3.1.2 with a few changes at the end. We now obtain $\varphi \in L$ such that

$$\int_{-\infty}^{\infty} f(x-\lambda_k)\hat{\varphi}^{\vee}(x)dx = 0.$$

Since $\hat{\varphi} \in L^{\infty}$ and $\hat{\varphi}$ is nontrivial the theorem is proved.

In the case of approximation in C_0 the theorems are quite similar to those we proved for L^2 and L.

Theorem 3.1.6 Let $\theta \in A$, let $f \in C_0$ be the Fourier transform of $g \in L$ such that $g(t) = O(e^{-\theta(t)})$, and suppose that g is not zero a.e. on any interval. If $\Sigma \frac{1}{\lambda_k^*} = \infty$ then $\Lambda(f)$ is dense in C_0.

Proof: A linear functional on C_0 is represented by a complex measure on R. Suppose $d\mu \perp \Lambda(f)$. By Fubinis theorem

$$(3.1.7) \qquad (\hat{f} * d\mu)(x) = \int \check{f}(x-y)d\mu(y) = \iint e^{-itx+ity}g(t)dt\, d\mu(y) =$$

$$= \int e^{-itx}g(t)\hat{\mu}(-t)dt.$$

Using theorem 2.1.2 we obtain $\hat{f} * d\mu \equiv 0$. By the inversion theorem for Fourier transforms $g(t)\hat{\mu}(-t) = 0$ a.e. Since $\hat{\mu}$ is continuous $\hat{\mu} \not\equiv 0$ would imply $g = 0$ a.e. on some open interval. This contradicts the hypothesis.

Theorem 3.1.7 Let $\theta \in B$, let $f \in C_0$ be the Fourier transform of $g \in L$, such that

$$(1/\hat{f}(t))\exp(-\theta(t)) \in L^2.$$

Suppose that $\hat{f}$ is differentiable and that

$$|Df/\hat{f}| \leq M.$$

If $\Sigma \frac{1}{\lambda_k^*} = \infty$, then $\Lambda(f)$ is not dense in C_0.

Proof: We follow the proof of theorem 3.1.2 with the modification $g_0(z) = g(z)/(z^4+2\alpha^4)$. Then both $\hat{g}_0/\hat{f}^\vee$ and $D\hat{g}_0/\hat{f}^\vee$ belong to L^2. Putting $\varphi = \hat{g}_0/\hat{f}^\vee$ we have

$$D\varphi = (D\hat{g}_0/\hat{f}^\vee)+(\hat{g}_0/\hat{f}^\vee)(Df^\vee/f^\vee) \in L^2.$$

By Carlson's inequality $\hat{\varphi} \in L$, hence $\hat{\varphi}^\vee(x)dx$ yields a nontrivial measure, orthogonal to $\Lambda(f)$.

Remark: It is easily seen that theorem 3.1.3 also has analogues in L and C_o.

3.2. Functions whose translates approximate exceptionally well

In this section we give two theorems showing that some regularity condition on $\hat{f}$ is necessary in theorem 3.1.7. since,in some sense, irregular behavior by $\hat{f}$ can imply good approximation properties in C_0. We also treat a related L^2-case.

We introduce $\Lambda(x,L)$ to be the number of λ_k:s in $[x-L,x+L]$ and suppose $\lambda_k < \lambda_{k+1}$, $(-\infty < k < \infty)$, $\lambda_0 = 0$.

Theorem 3.2.1.

Let f be the Fourier transform of g, and suppose

(a) $g(t) = O(e^{-\alpha|t|})$ for some $\alpha > 0$,

(b) the set of points where g:s left and right hand limits exist and are different has a finite accumulation point, ξ,

(c) g is not zero a.e. on any open interval.

Then $\sum_{\lambda_k \in \Lambda} (1+\lambda_k^2)^{-1} = \infty$ implies that $\Lambda(f)$ is dense in C_0 if Λ satisfies the regularity condition:

(d) for some function φ such that $\varphi(e^x)$ is convex and some positive constants c_1, c_2, L,

$$c_1 \cdot \varphi(|x|) \leqq \Lambda(x,L) \leqq c_2 \cdot \varphi(|x|).$$

Remark. The regularity condition on Λ demands, among other things, that the positive and negative λ_k:s behave similarly.

Proof Suppose $d\mu \perp \Lambda(f)$ and extend $F = \check{f} * d\mu$ analytically to S_α. F will be uniformly bounded in $S_{\alpha/2}$ By "Schwarz's lemma"

$$\log |F(x)| \leqq \Sigma \log \left| \frac{\exp(\pi x/\alpha)-\exp(\pi\lambda_k/\alpha)}{\exp(\pi x/\alpha)+(\exp(\pi\lambda_k/\alpha)} \right| + C \leqq$$

$$\leqq -2\Sigma \exp(-\frac{\pi}{\alpha}|x-\lambda_k|) + C \leqq$$

$$\leqq - C' \Lambda(x,L) + C \leqq - C'' \varphi(|x|) + C. \qquad (C''>0)$$

Since

$$\int_0^\infty \frac{\varphi(x)\,dx}{1+x^2} \geqq \frac{1}{2c_2} \int_1^\infty \frac{\Lambda(x,L)}{x^2}\,dx \geqq$$

$$\geqq \frac{1}{2c_2} \int_1^\infty \int_{x-L}^{x+L} dn(\lambda)\,\frac{dx}{x^2} \geqq \frac{1}{2c_2} \sum_{\lambda_k \geqq 1+L} \left(\int_{\lambda_k-L}^{\lambda_k+L} \frac{dx}{x^2} \right) =$$

$$= \frac{L}{c_2} \sum \frac{1}{\lambda_k^2 - L^2} = \infty$$

and since $\varphi(e^x)$ is convex it follows by a theorem by Izumi and Kawata [9] that $\hat{F}$ belongs to a quasi-analytic class.

If we can show $\hat{F}^{(n)}(\xi)=0, \forall n$, then $\hat{F}\equiv 0$ follows and we obtain $d\mu=0$ just as in theorem 3.1.6.

By (3.1.7) and since $F \in L$

$$\hat{F}(t) = g(t)\,\hat{\mu}(-t) \quad \text{a.e.} \tag{3.2.1}$$

Let t_i be a discontinuity point of g as described in (b) and let t tend to t_i avoiding points where (3.2.1) does not hold. Since $\hat{F}$ and $\hat{\mu}$ are continuous, we have

$$\hat{\mu}(-x_i) \lim_{x\uparrow x_i} g(x) = \hat{\mu}(-x_i) \lim_{x\downarrow x_i} g(x) = \hat{F}(x_i).$$

But the limits are different, hence $\hat{\mu}(-x_i) = 0$ and $\hat{F}(x_i) = 0$.

Since $x_i \to \xi$ it follows by repeated application of Rolle's theorem that $\hat{F}^{(n)}(\xi) = 0$, $\forall n$, and the theorem is proved.

An example showing that the condition $\Sigma (1+\lambda_k^2)^{-1} = \infty$ is sharp will be given at the end of this section.

It is not necessary that the discontinuities of g accumulate at finite point.

Theorem 3.2.2 Let f be the Fourier transform of g and suppose

(a) $g(t) = O(e^{-\alpha|t|})$ for some $\alpha > 0$,

(b) there is a strictly increasing sequence of points t_k, such that the right and left hand limits of g at t_k exist and are different and

$$\liminf_{k \to \infty} t_k k^{-1/p} < \infty, \quad (p > 1),$$

(c) g is not zero a.e. on any open interval.

If, for some $L > 0$, $\Lambda(x,L)/|x|^q \to \infty$ as $x \to \pm\infty$, then $\Lambda(f)$ is dense in C_0 (q is the conjugate exponent to p).

Proof. Estimating $|F(x)|$ in the same way as in the proof of theorem 3.2.1, we obtain for each $A > 0$ a constant $C_A > 0$, such that

$$|F(x)| \leqq C_A \exp(-A|x|^q).$$

Since $q > 1$ it is possible to extend $\hat{F}$ analytically to an entire function. Then

$$|\hat{F}(x+iy)| \leqq C_{A+1} \int_0^\infty \exp(-t^q) \exp(t|y| - At^q)\, dt \leqq$$

$$\leqq C' \exp((q-1)q^{-p} A^{1-p}|y|^p) = C' \exp(B_A|y|^p),$$

where

$$B_A \to 0 \quad \text{as} \quad A \to \infty .$$

Proceeding as in theorem 3.2.1 we conclude $\hat{F}(t_k) = 0$. Let $n(r)$ denote the number of zeros of $\hat{F}$ in $[-r,r]$. By hypothesis

$$\limsup_{r \to \infty} \left(\frac{n(r)}{r^p}\right) \geqq \limsup_{k \to \infty} \left(\frac{k}{t_k^p}\right) = \tag{3.2.2}$$

$$= \left(1/\liminf_{k \to \infty} t_k k^{-1/p}\right) > 0.$$

Since $\hat{F}$ is of order p and type B_A we know that

$$\limsup_{r \to \infty} \left(\frac{n(r)}{r^p}\right) \leqq ep\, B_A$$

if $\hat{F}$ is not indentically zero ([4]p.16). Hence we have a contradiction if A is chosen large enough. Thus $\hat{F} \equiv 0$ and the rest follows as for theorem 3.1.6.

Example: By appropriate choice of coefficients β_k the function

$$g(t) = \sum_{k=0}^{\infty} \beta_k \chi[-t_k,t_k]$$

satisfies the conditions (a)-(c). Here χ is the indicator function. Taking Fourier transforms we obtain

$$f(x) = \sum_{k=0}^{\infty} \beta_k \frac{\sin(t_k x)}{x}$$

In particular, if $t_k = k^2$ and $\beta_k = e^{-k^2}$,

$$g(t) = \sum_{t_k \geqq |t|} \beta_k = \sum_{k^2 \geqq |t|} \beta_k \leqq e^{-|t|} + \ldots \leqq C\, e^{-|t|}, \quad C > 0.$$

On the other hand

$$\sum_{k^2 \geqq |t|} \beta_k \geqq \sum_{k \geqq [\sqrt{|t|}]+1} \geqq e^{-|t|-2\sqrt{|t|}-1} .$$

By theorem 3.1.3 $\Lambda(f)$ is not dense in L^2 if $\exp(-\frac{\pi}{2}|\lambda_k|) < \infty$. In C_0, however, by theorem 3.2.2 $\Lambda(f)$ is dense if $\Lambda = \{\lambda_k : -\infty < k < \infty\}$ where $\lambda_0 = 0$, $\lambda_k < \lambda_{k+1}$ and $|\lambda_k| = o(|k|^{1/3})$ as $|k| \to \infty$.

In our next theorem the zeros of f play the role of the discontinuities in the preceeding theorem.

Theorem 3.2.3 Let f be the Fourier transform of g such that $g(t) = O(e^{-\alpha|t|})$ for some $\alpha > 0$, $(|t| \to \infty)$. Suppose that g is differentiable and that there is a strictly increasing sequence of zeros of g, $\{t_k\}_0^\infty$, such that

$$\liminf_{k \to \infty} (t_k k^{-1/p}) < \infty .$$

If $\Lambda(x,L)/|x|^q \to \infty$ as $|x| \to \infty$ for some $L > 0$ then $\Lambda(f)$ is dense in L^2 and C_0.

Proof. We take the L^2-case first showing that $\hat{F}(t_k) = 0$ ($F = f * \check{\varphi}$, $\varphi \in L^2$, $\varphi \perp \Lambda(f)$).

Almost everywhere

$$\hat{F}(t) = g(t)\hat{\varphi}(t). \tag{3.2.3}$$

Letting t tend to t_k, avoiding the set of points where (3.23) does not hold, we have, since g is differentiable and $\hat{F}$ continuous,

$$|\hat{F}(t_k)| = \lim_{t \to t_k} |g(t)\hat{\varphi}(t)| \leq \lim_{t \to t_k} |t-t_k| \cdot |\hat{\varphi}(t)| \cdot C, \quad C > 0.$$

If $\hat{F}(t_k) \neq 0$, then $|\hat{\varphi}(t)| > C_0/|t-t_k|$ a.e. in some neighbourhood of t_k. This contradicts $\hat{\varphi} \in L^2$, hence $\hat{F}(t_k) = 0$.

In the C_0-case it is easy to see that $\hat{F}(t_k) = 0$. For the remaining part of the proof we refer to the proof of theorem 3.2.2.

Example: We show that the condition $\Sigma (1+\lambda_k^2)^{-1} = \infty$ in thm 3.2.1 is sharp:

If the series converges we can find G, holomorphic and uniformly bounded in the half plane $\{z: \mathrm{Im}(z) < \alpha'\}$, $\alpha' > \alpha$ with $G(\lambda_k) = 0$.

Put

$$H(x) = \frac{G(x)}{(x-i\beta)^5} . \qquad (\beta > \alpha')$$

By Paley-Wiener's theorem

$$\hat{H}(t) = 0 , \quad t > 0.$$

Since $x^3 H(x) \in L$ if follows that $\hat{H}$ is twice continuously differentiable, hence $\hat{H}(t)/t^2$ is bounded in a neighbourhood of $t = 0$. It is also easy to see that $\hat{H}(t) = O(\exp(\alpha' t))$ as $t \to -\infty$

Put

$$g(t) = \hat{H}(t)\left(\frac{1+t^4}{t^2}\right) , \quad t < 0$$

and extend g to R so that (a), (b), and (c) are satisfied.

Put

$$\varphi(t) = \begin{cases} t^2/1+t^4 , & t < 0, \\ 0 , & t \geq 0. \end{cases}$$

Since $H = \hat{g}^\vee * \hat{\varphi}^\vee = \check{f} * \hat{\varphi}^\vee$ and $\hat{\varphi}^\vee \in L$ is nontrivial we find that $\Lambda(f)$ is not dense in C_0.

3.3. Quasi-analyticity.

We use a theorem of Hirschman [8] to obtain a result for functions in certain quasi-analytic classes. Only the case $\alpha = 0$ in thm 3.3.1 is of interest since for $\alpha > 0$ we can use either thm 3.1.1 or, if Λ and θ permit, thm 2.3.1 to obtain stronger conclusions.

Suppose that ν is a continuously differentiable function satisfying

(a) $\nu(0) = 1$,

(b) $\nu'(x) \geqq 0$, $x \geqq 0$,

(c) $x\nu'(x)/\nu(x) \to 0$ as $x \to \infty$.

Let $M_n = n!(\nu(n))^n$. By $f \in C\{M_n, k\}$ we mean $|f^{(n)}(t)| \leqq A k^n M_n$, $-\infty < t < \infty$, for all n and some constant A. Define H by

$$H(v) = \frac{2}{\pi} \int_1^v \frac{\log T(t)}{t^2}\, dt$$

where $T(t) = \max_{n \geqq 0} (\frac{t^n}{M_n})$. We suppose $H(v) \to \infty$ as $v \to \infty$. Hirschmans theorem reads:

Theorem. Let $f \in C\{M_n, k\}$, $f \not\equiv 0$. If $Z(t)$ is the number of zeros of f in the interval $[-t,t]$ counted according to their multiplicities, then

$$\limsup_{t \to \infty} H(Z(t))/t \leqq k.$$

We now prove the following theorem

Theorem 3.3.1. Let $f \in L^2$ satisfy $\hat{f}(t) = O(e^{-\theta(t)})$ where θ is an even, positive, continuously differentiable function, concave for $x \geqq 0$, such that

(i) $\theta(x)/x \to \alpha \geqq 0$ as $x \to \infty$,

(ii) $x\theta'(x)/\theta(x) \to 1$ as $x \to \infty$,

(iii) $\theta(0) = 0,\ \theta'(0) = 1,$

(iv) $\int_1^\infty \frac{\theta(x)}{x^2}\,dx = \infty.$

If $\hat{f}(t) \neq 0$ a.e. and

(3.3.1) $$\limsup_{k\to\infty} \{\frac{2}{\pi\lambda_k} \int_1^k \frac{\theta(x)}{x^2}\,dx\} > 1$$

then $\Lambda(f)$ is dense in L^2. $(\lambda_{k+1} > \lambda_k > 0)$.

Proof: Put $\nu(x) = \theta^{-1}(x)/x$ for $x \geq 0$. By elementary calculations it follows that ν satisfies the conditions (a) - (c). For $t \geq 0$ we denote by $n(t)$ the least n for which $\max(t^n/M_n)$ is attained.

Following Hirschman we find

(3.3.2) $$\log T(t) \sim n(t),$$

(3.3.3) $$n(t)\,\nu(n(t)) \sim t,$$

where $f_1(t) \sim f_2(t)$ means that $f_1(t)/f_2(t) \to 1$ as $t \to \infty$. Observe that that $t \sim s$ implies $\theta(t) \sim \theta(s)$ since, for $x \geq y$,

$$\left|\log\left(\frac{\theta(x)}{\theta(y)}\right)\right| = \int_y^x \frac{\theta'(t)}{\theta(t)}\,dt \leq \int_y^x \frac{dt}{t} = \left|\log\left(\frac{x}{y}\right)\right|.$$

Hence, by (3.3.3) and the definition of θ,

$$\theta(t) \sim \theta(n(t)\,\nu(n(t))) = n(t).$$

By (3.3.2)

$$\theta(t) \sim \log T(t).$$

If we now take $\varphi \in L^2$, $\varphi \perp \Lambda(f)$ and put $F = \overset{\vee}{f} * \varphi$ we have $\hat{F} = \hat{f}^{\vee}\hat{\varphi}$. Given $\varepsilon > 0$ there is $A > 0$ such that

$$|\hat{f}(t)| \leq e^{-\theta(t)} \leq A(T(t))^{-(1-\varepsilon)}.$$

Estimating the derivatives of F by Fouriers inversion formula we have

$$|F^{(n)}(x)| \leq A' \int_1^\infty t^n (T(t))^{-(1-\varepsilon)}\,|\hat{\varphi}(t)|\,dt + O(1) \leq$$

$$\leq A'' \max_{t\geq 0} \{t^{n+1}(T(t))^{-(1-\varepsilon)}\} \text{ , since } \hat{\varphi} \in L^2.$$

Using the inequality $\max_{t\geq 0} \frac{t^n}{T(t)} \leq M_n$ and the properties of ν we find (see [8] for the details)

$$|F^{(n)}(x)| \leq Cn!\,(\nu(n))^n\,(1-\varepsilon)^{-4n}.$$

Hence $F \in C\{M_n, (1-\varepsilon)^{-4n}\}$. We apply Hirschman's theorem. Since $\varepsilon > 0$ can be chosen arbitrarily small

$$\limsup_{t\to\infty} H(Z(t))/t \leq 1 \quad \text{if} \quad F \not\equiv 0.$$

But (3.3.1) implies that the lim sup exceeds 1, since $F(\lambda_k) = 0$. Hence $F \equiv 0$ and $\varphi = 0$ a.e. and the theorem is proved.

Remark: We conjecture that $\sum \frac{1}{\lambda_k^*} = \infty$ can replace the lim sup condition.

In the following two theorems Λ denotes an arbitrary subset of R.

Theorem 3.3.2. Let $f \in L^2$, $\hat{f}(t) \neq 0$ a.e. and suppose that $|\hat{f}(t)| \leq$ $\leq C \exp(-\theta(t))$, where θ is increasing for $t \geq 0$ and

$$\int_1^\infty \frac{\theta(t)}{t^2}\,dt = \infty.$$

If the closure of Λ has positive Lebesgue measure, then $\Lambda(f)$ is dense in L^2.

Proof. Take $\varphi \in L^2$, $\varphi \perp \Lambda(f)$. Put $F = \overset{\vee}{f} * \varphi$ and $d\mu(t) = \hat{f}^{\vee}(t)\,\hat{\varphi}(t)dt$. The Fourier transform of $d\mu$ is $\overset{\vee}{F}$. Put

$$s(x) = \int_x^\infty |d\mu| = \int_x^\infty |\hat{f}^{\vee}(t)|\,|\hat{\varphi}(t)|\,dt \leq$$

$$\leq C \int_x^\infty \exp(-\theta(t))|\hat{\varphi}(t)|\,dt \leq C \exp(-\frac{\theta(x)}{2}) \int_x^\infty \exp(-\frac{\theta(t)}{2})|\hat{\varphi}(t)|\,dt \leq$$

$$\leq C' \exp(-\frac{\theta(x)}{2}). \quad (x \geq 0).$$

We use the following result of Beurling [2]:

If g is the Fourier transform of a bounded measure $\mu(t)$, $s(\lambda) = \int_{\lambda}^{\infty} |d\mu(t)|$, $\int_{1}^{\infty} \log s(\lambda) \frac{d\lambda}{\lambda^2} = -\infty$ and g is zero on a set of positive measure, then $g \equiv 0$.

Since $\overset{\vee}{F}$ is zero on $\overline{\Lambda}$ (continuity) and $\int_{1}^{\infty} \log s(x) \frac{dx}{x^2} \leqq$

$\leqq -\frac{1}{2} \int_{1}^{\infty} \frac{\theta(x)}{x^2} dx + C'' = -\infty$ we conclude $F \equiv 0$ and $\varphi = 0$ a.e.

Theorem 3.3.3. Let $f \in L^2$ and suppose $|\hat{f}(t)| \geqq C \exp(-\theta(t))$, where θ is even, positive, increases with $|t|$ and

$$\int_{1}^{\infty} \frac{\theta(x)}{x^2} dx < \infty.$$

If Λ is not dense in R, then $\Lambda(f)$ is not dense in L^2.

Proof: N. Levinson has proved, [12] p. 81, that for such a θ there is, given $x_0 > 0$, a function $H \not\equiv 0$, such that

$$|H(t)| \leqq \frac{\exp(-\theta(t))}{1+t^2},$$

and $\hat{H}(0) \neq 0$, $\hat{H}(x) = 0$ for $|x| > x_0$.

Assume $[y-x_0, y+x_0] \cap \Lambda$ is empty. Put

$$g(t) = e^{-iyt} H(t)/\hat{f}(t).$$

Then $g \in L^2$ and $\hat{f}^{\vee}(t)g(t) = e^{-iyt} H(t)$ implies

$$(\check{f} * \hat{g}^{\vee})(x) = \hat{H}(x-y).$$

Since $\hat{H}(x) = 0$ for $|x| > x_0$, $\check{f} * \hat{g}^{\vee}$ is zero on Λ, hence $\hat{g}^{\vee}$ is a nontrivial element of L^2, orthogonal to $\Lambda(f)$. This proves the theorem.

3.4. Entire functions and sufficient sets of translation numbers.

Let us call a set Λ sufficient for L-approximation if, for some $f \in L$, $\Lambda(f)$ is dense in L. We have seen that if f has a rapidly decreasing Fourier transform, $\Lambda(f)$ can be dense even if Λ is quite thin. We will prove some theorems about sufficient sets in this section.

Before that we deal with approximation by translates of functions whose Fourier transforms decrease faster than exponentially, more precisely we require $f(t) = 0(\exp(-\alpha|t|^p))$, $p > 1$. In the general case one can use Jensen's formula to obtain similar theorems but they will not be very sharp and we omit this.

Theorem 3.4.1. Let $f \in L^2$, suppose $\alpha > 0$, $p > 1$ and $\hat{f}(t) = 0(\exp(-\alpha|t|^p))$, $\hat{f}(t) \neq 0$ a.e. Let q be the conjugate exponent to p and suppose that for some $\delta > 0$

$$\lambda_{n+1}^q - \lambda_n^q \geq \delta, \qquad (\lambda_n > 0)$$

and

$$\limsup_{r \to \infty} \{ \sum_{0<\lambda_n<r} \lambda_n^{-q} - \frac{(p\alpha)^{1-q}}{\pi} (\sin(\frac{\pi}{2q}))^q \log r\} = \infty.$$

Then $\Lambda(f)$ is dense in L^2.

In the proof we use the following lemma.

Lemma 3.4.2. Suppose G is an entire function such that

$$\limsup_{r\to\infty} \frac{\log|G(re^{i\theta})|}{r^q} \leq \beta|\sin\theta|^q,$$

$q > 1$, $\beta > 0$. Suppose that for $n \geq 0$, $\lambda_n > 0$, $\lambda_{n+1}^q - \lambda_n^q \geq \delta > 0$, and $G(\lambda_n) = 0$. If

$$\limsup_{r \to \infty} \{ \sum_{\lambda_n<r} \lambda_n^{-q} - \frac{\beta q}{\pi} (\sin(\frac{\pi}{2q}))^q \log r\} = \infty,$$

then $G \equiv 0$.

Proof. Define $H(z) = G(z^{1/q})$ for $|\arg(z)| \leq \frac{\pi}{2}$. Then

$$\limsup_{r\to\infty} \frac{\log|H(re^{i\theta})|}{r} = \limsup_{r\to\infty} \frac{\log|G(r^{1/q}e^{i(\theta/q)})|}{r} \leq$$

$$\leq \beta(\sin(\theta/q))^q \leq \beta(\sin(\tfrac{\pi}{2q}))^q = c, \text{ say.}$$

We now use Fuchs' theorem concerning zeros of functions of exponential type, [6]. Evidently H is of exponential type c, has zeros at the points $\mu_n = \lambda_n^q$, $\mu_{n+1} - \mu_n \geq \delta > 0$ and

$$\limsup_{R\to\infty}\{\sum_{\mu_n<R} \frac{1}{\mu_n} - \frac{c}{\pi}\log R\} = \limsup_{r\to\infty}\{\sum_{\lambda_n<r} \lambda_n^{-q} - \frac{cq}{\pi}\log r\} = \infty.$$

Hence $H \equiv 0$.

Proof of theorem 3.4.1. With $\varphi \perp \Lambda(f)$ and $F = \overset{\vee}{f} * \varphi$, F will be entire and

$$|F(x+iy)| \leq C\int_0^\infty \exp(-\alpha t^p + \log(1+t^2) + |y|t)\ \frac{\hat{\varphi}(t)dt}{1+t^2} \leq$$

$$\leq C' \exp(\max_{t\geq 0}\{y|t| - \alpha t^p + \log(1+t^2)\}).$$

If the maximum is attained at $t = \tau$, then

$$|y| = \alpha p\tau^{p-1} - \frac{2\tau}{1+\tau^2},$$

hence $|y| + 1 \geq \alpha p\tau^{p-1} \geq |y|$ and the maximum value is

$$|y|\tau - \alpha\tau^p + \log(1+\tau^2) \leq \alpha(p-1)\tau^p + \log(1+\tau^2) \leq$$

$$\leq \alpha^{1-q}(p-1)p^{-q}(|y|+1)^q + o(|y|).$$

Hence

$$\limsup_{r\to\infty} \left(\frac{\log|F(re^{i\theta})|}{r^q}\right) \leq \alpha^{1-q}(p-1)p^{-q}(|\sin\theta|)^q.$$

Now $F \equiv 0$ follows from lemma 3.4.2, since $F(\lambda_n) = 0$, and $\alpha^{1-q}(p-1)p^{-q}q = (p\alpha)^{1-q}$. It then follows that $\varphi = 0$ a.e. and the theorem is proved.

The coefficient for $\log r$ in the lemma is best possible. This follows from the existence of nontrivial entire functions of order q and type β, satisfying the conditions of the lemma with $\lambda_n = a\,n^{1/q}$, where

$$a = (\frac{\pi}{\beta})^{1/q} \frac{1}{\sin(\frac{\pi}{2q})}.$$

The lemma shows that a cannot be larger. We shall give an example of such a function for q an even number; the construction for general $q > 1$ can be based on the same inequality, but we have not found any explicit formulas.

Lemma 3.4.3. Suppose $q \geq 1$. For $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$,

$$(3.4.1) \qquad (\sin(\frac{\pi}{2q}))^{q-1}(-\cos(q\theta + \frac{\pi}{2q})) \leq |\sin\theta|^q.$$

Proof. Let $\theta \geq 0$. Denote the left member of (3.4.1) by $x(\theta)$ and the right member by $y(\theta)$. Note that $x(\frac{\pi}{2q}) = y(\frac{\pi}{2q})$ and $x'(\frac{\pi}{2q}) = y'(\frac{\pi}{2q})$. The maximum of x is attained when $q\theta + \frac{\pi}{2q} = \pi$ and the minimum when $q\theta + \frac{\pi}{2q} = 0$, i.e. at

$$\theta_1 = \frac{\pi}{2}(1 - (\frac{q-1}{q})^2) < \frac{\pi}{2}$$

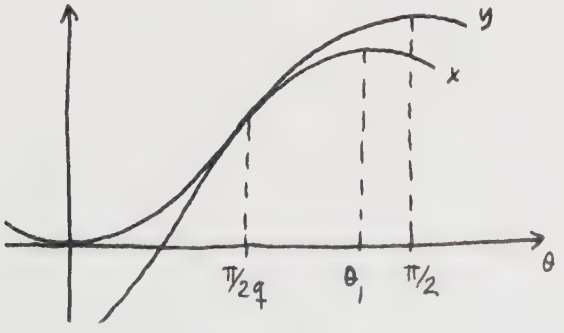
and

$$\theta_2 = -\frac{\pi}{2q^2} < 0$$

respectively, and x is increasing in between.

It is sufficient (in the case $\theta \geq 0$) to prove the inequality for $\theta \in I$ = the interval where $x > 0$ and $\theta \leq \theta_1$ (see figure 6).

Figure 6.



Taking second derivatives we have $x'' = -q^2 x$ and $y'' \geq -q^2 y$. Multiplying by y and x respectively and subtracting yields

$$xy'' - x''y \geqq 0, \qquad \theta \in I.$$

Hence $D(xy'-x'y) \geqq 0$, and since $xy' - x'y = 0$ for $\theta = \frac{\pi}{2q}$ we obtain

$$D(\frac{x}{y}) = \begin{cases} \geqq 0 & \text{if } \theta \geqq \frac{\pi}{2q} \\ \leqq 0 & \text{if } \theta \leqq \frac{\pi}{2q} . \end{cases}$$

Hence $y \geqq x$ for $\theta \in I$.

To prove 3.4.1 for $\theta \leqq 0$ we show that for $\theta \geqq 0$,

$$(3.4.2) \qquad (\sin(\frac{\pi}{2q}))^{q-1}(-\cos(-q\theta + \frac{\pi}{2q})) \leqq (\sin\theta)^q.$$

Denote the left member by $z(\theta)$. It is geometrically evident that for $0 \leqq q\theta \leqq \pi$

$$\cos(q\theta + \frac{\pi}{2q}) \leqq \cos(-q\theta + \frac{\pi}{2q})$$

with equality for $\theta = 0$ and $\theta = \pi/q > \theta_1 = \frac{\pi}{q}(1 - \frac{1}{2q})$. Thus (3.4.2) holds for $\theta \leqq \frac{\pi}{q}$. For $\theta > \frac{\pi}{q}$ it follows from the first part of the proof that $y(\theta) > x(\theta_1) = \max z(\theta)$. The lemma is proved.

Now let

$$F_q(z) = \sin\{(\sin(\frac{\pi}{2q}))^q z^q\} \exp\{-\cos(\frac{\pi}{2q})(\sin(\frac{\pi}{2q}))^{q-1} z^q\},$$

where q is an even number. F_q can be written as

$$\frac{i}{2}\{\exp(-Az^q) - \exp(-\bar{A}z^q)\}$$

where $A = (\cos(\frac{\pi}{2q}) + i\sin(\frac{\pi}{2q}))(\sin(\frac{\pi}{2q}))^{q-1}$.

Then

$$|\exp(-Az^q)| = \exp\{\mathrm{Re}(-(\sin(\tfrac{\pi}{2q}))^{q-1} e^{i(q\theta + \frac{\pi}{2q})} r^q)\} =$$

$$= \exp(r^q(\sin\tfrac{\pi}{2q})^{q-1}(-\cos(q\theta + \tfrac{\pi}{2q}))) \leqq \exp(|r\sin\theta|^q)$$

for $|\theta| \leqq \frac{\pi}{2}$ by lemma 3.4.3. Since q is even, $F_q(z) = F_q(-z)$, hence the estimate is valid for all θ.

The second term can be estimated in the same way. Then we have

$$|F_q(re^{i\theta})| \leqq \exp(r^q|\sin\theta|^q),$$

and $F_q(\lambda_n) = 0$ where

$$\lambda_n = \pi^{1/q} \frac{1}{\sin(\pi/2q)} n^{1/q}.$$

We now turn to some theorems concerning sufficient sets of translation numbers.

Theorem 3.4.4. Suppose $\{\lambda_n\}_1^\infty$ is a strictly increasing sequence such that

$$\liminf_{n\to\infty} \left(\frac{\lambda_n}{n}\right) = 0.$$

Then there is $f \in L$ such that $\Lambda(f)$ is dense in L.

Proof. We can assume $0 \notin \Lambda$. Let $n(t)$ be the number of λ_k in $[-t,t]$. Choose a sequence of points t_k such that $t_k \uparrow \infty$ and $n(t_k)/t_k \uparrow \infty$ as k tends to infinity.

Put $N(t) = \int_0^t \frac{n(s)}{s}\, ds$.

Since

$$\frac{1}{t_k} \int_{t_k/e}^{t_k} \frac{n(s)}{s} ds \geq \frac{n(t_k)}{t_k}$$

we have $N(t_k)/t_k \uparrow \infty$, hence $\dfrac{N(t_{k+1}) - N(t_k)}{t_{k+1} - t_k} \uparrow \infty$. Let ν be the piecewise linear function with nodes at $t = t_k$, such that $\nu(t_k) = N(t_{k-1})$, $k \geq 1$, and $\nu(0) = N(t_0)$. It is convex, increasing and, by construction, $\nu(t) \leq N(t)$.

We can assume $t_k - t_{k-1} \geq 1$. Hence

$$N(t_k) - \nu(t_k) = N(t_k) - N(t_{k-1}) \uparrow \infty. \tag{3.4.3}$$

Let f be the function whose Fourier transform is

$$\hat{f}(t) = \int_{|t|}^\infty \exp(-\nu^*(s)) \frac{ds}{1+s^2},$$

(ν^* the convex conjugate to ν).

Then $|\widehat{Df}(t)| = \dfrac{\exp(-\nu^*(|t|))}{1+t^2}$, $(t \neq 0)$

and $\hat{f}(t) = O(\frac{\exp(-\nu^*(|t|))}{1+t^2})$.

Putting $F = \check{f} * \varphi$ for some $\varphi \in L^\infty$ such that $\varphi \perp \Lambda(f)$ we have, as in the proof of theorem 3.1.4, $|F(x+iy)| \leq C \exp(\nu(|y|))$. If $F \not\equiv 0$ it is no restriction to assume $F(0) \neq 0$.

By Jensen's formula applied to F,

$$N(t) \leq \nu(t) + O(1).$$

This contradicts (3.4.3) and we conclude $F \equiv 0$. Hence $\varphi = 0$ a.e. since $\hat{f}(t) \neq 0$, $\forall t$. The theorem is proved.

One cannot go much further. The following theorem originates from joint work with B. Johansson, who has used a quantitative version to study "poor approximability", [10].

Theorem 3.4.5. Let $f \in L$. If $\lambda_{n+1} - \lambda_n \geq c > 0$, $\forall n \in Z$, then $\Lambda(f)$ is not dense in L.

Proof. We can assume $\hat{f}(t) \neq 0$, $\forall t$, since otherwise the assertion is the converse part of Wiener's approximation theorem. Assume further that $0 \notin \Lambda$. We will show that f does not belong to the closure of $\Lambda(f)$ in L.

In the proof we will suppose that $|\lambda_n - cn| \leq \frac{c}{2}$ for all $n \neq 0$. One can adjoin the number nc to Λ if there is no λ in the interval $](n-\frac{1}{2})c, (n+\frac{1}{2})c]$. If the assertion of the theorem holds for the larger Λ it a fortiori holds for the original Λ.

Put $F(z) = (\frac{\sin z}{z})^m \prod_{-\infty}^{\infty} (1 - \frac{z^2}{\lambda_n^2})$.

It is well known (see e.g. [15]) that F is an entire function of exponential type and that m can be chosen so that

$$|F(x)| \leq C \frac{1}{1+x^2}.$$

Paley-Wieners second theorem shows that $\hat{F}$ has compact support, K say. By Wiener's lemma $1/\hat{f}$ coincides on K with $\hat{\varphi}$, the Fourier transform of $\varphi \in L$. Thus $\hat{F}(1/\hat{f}) = \hat{F} \cdot \hat{\varphi}^{\vee}$.

Put $\Psi = F * \varphi$. Then $\Psi \in L$ and $\hat{\Psi} = \hat{F}\,\hat{\varphi}^{\vee}$ and it follows that $\hat{\Psi}\hat{f} = \hat{F}\,\hat{\varphi}^{\vee}\hat{f}^{\vee} = \hat{F}$.

Hence $F = \check{f} * \Psi$, $\Psi \in L^{\infty}$, $F(\lambda_n) = 0$, $F(0) \neq 0$ which proves the assertion of the theorem.

It is easy to see that theorem 3.4.4 has an analogue in L^2, the proof is almost the same. To prove an analogue to theorem 3.4.5, however, one cannnot copy the proof since division by $\hat{f}$ on an arbitrary compact set will in general not give a L^2-function.

Theorem 3.4.6 Let $f \in L^2$ and let $\Lambda = \{c \cdot n, n \in Z\}$ for some $c > 0$. $\Lambda(f)$ is not dense in L^2.

Proof. We can assume $\hat{f}(t) \neq 0$ a.e. We will construct $\varphi \in L^2$ and a set A, such that $m(A) > 0$, (Lebesgue measure) and

$$\chi_A(t) - \chi_A\left(t - \frac{2\pi}{c}\right) = \check{\hat{f}}(t)\varphi(t). \tag{3.4.4}$$

Taking Fourier transforms we have

$$\left(1 - \exp\left(\frac{-2\pi i x}{c}\right)\right) \hat{\chi}_A(x) = (\check{f} * \hat{\varphi})(x).$$

Hence $\hat{\varphi} \perp \Lambda(f)$ and since $\hat{\varphi}$ is nontrivial and belongs to L^2, this will prove the theorem.

Put $E_y = \{t: |\hat{f}(t)| \geq y\}$ for $y > 0$.

If we choose y small and put $a = \frac{2\pi}{c}$, then $m(E_y \cap (0,a)) \geq \frac{3a}{4}$ and $m(E_y \cap (a,2a)) \geq \frac{3a}{4}$.

Define $A = \{t: 0 < t < a,\ t \in E_y,\ t+a \in E_y\}$. The measure of the complement of A relative to $(0,a)$ is at most $2(\frac{a}{4})$ hence $m(A) > 0$.

Now $\chi_A(t) - \chi_A(t-a)$ is different from zero only if $t \in A$ or $t - a \in A$. In both cases $t \in E_y$ by the definition of A.

Put $$\varphi(t) = \frac{\chi_A(t) - \chi_A(t-a)}{\overset{\wedge\vee}{f}(t)}$$

Evidently $|\varphi(t)| \leq \frac{1}{y}$ and φ has compact support. Hence $\varphi \in L^2$ and the construction of φ and A is complete.

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